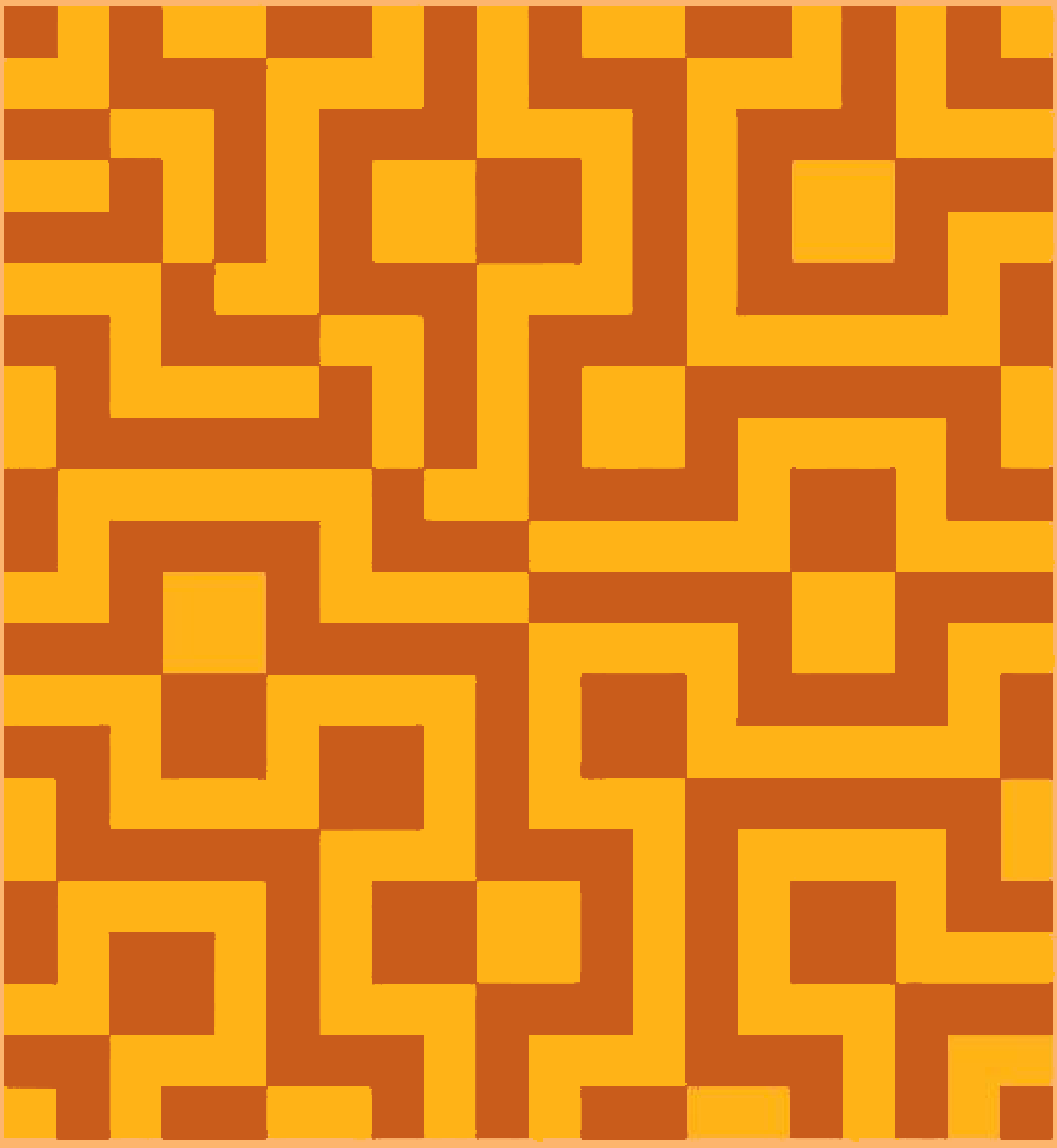
Paulus Gerdes

Lunda Geometry



Mirror curves, Designs, Knots, Polyominoes, Patterns, Symmetries

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Universidade Pedagógica
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2007

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LUNDA GEOMETRY:

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PREFACE

(First edition)

Lunda-designs are a type of black-and-white design that I discovered while analyzing properties of a special class of *sona* sand drawings from eastern Angola and neighboring regions of Zambia and Congo / Zaire. The elements of this class are mirror-generated curves, and generate, in turn, Lunda-designs.

The chapters of this book are comprised of papers on mirror curves, Lunda-designs and related concepts such as Lunda-polyominoes and Lunda-patterns. Most chapters may be read independently of each other.

Chapter 1 is a partial translation of Chapter 6 of the second volume of *Sona Geometry*, and analyses the mirror curve class of *sona* sand drawings and some of their basic properties. In this chapter, I also describe the discovery of Lunda-designs. Chapters 2 and 3 present an introduction to Lunda-designs, their symmetries, and some of their generalizations such as Lunda-k-designs and hexagonal Lunda-designs. These papers were published in the international journals *Visual Mathematics* (1999) and *Computers and Graphics* (1997) respectively.

In Chapter 4, I introduce the concepts of Lunda-polyominoes and Lunda-animals, and evaluate the number of possible paths of given lengths that may be traversed by Lunda-animals. The famous sequence of Fibonacci surprisingly appears in this context.

Chapter 5 presents a first approximation for the number of Lunda-n-ominoes. In Chapters 6 and 7, I explore special classes of Lunda-polyominoes such as symmetrical closed Lunda-polyominoes and Lunda-spirals. Finally, in Chapter 8, I show that for all twenty-four classes of one-color and two-color, one-dimensional patterns, it is possible to construct Lunda-strip-patterns, which belong to them. Furthermore, I present examples of one-color and two-color, two-dimensional Lunda-patterns.

Appendix 1 presents the proof of the theorem that every mirror design generates a Lunda-design, and, inversely, for every (finite) Lunda-design a mirror design that generates it may be constructed.

Lunda-designs present a concrete example of how ethnomathematical research can lead to both fruitful and interesting ideas of serious mathematical reflection. I hope the book *LUNDA Geometry* will stimulate further research on these aesthetically attractive figures.

Acknowledgements

My thanks go to my colleagues at the *Universidade Pedagógica* (UP, Mozambique) for their permanent interest and encouragement of my research; to Jill Gerrish (Department of English, UP) for the linguistic revision.

I am very grateful for the interest that colleagues in various parts of the world have shown in the preparation of this book. My thanks go, in particular, to Slavik Jablan (The Mathematical Institute, Belgrade, Serbia), Maurice Bazin (Exploratorium, San Francisco, USA), Erhard Scholz (Bergische Universität, Wuppertal, Germany), Clifford Pickover (IBM Thomas Watson Research Center, New York, USA), and Arthur Powell (Rutgers University, Newark, USA) for their stimulating reactions on the chapters they have read.

Above all, I thank my wife Marcela Libombo and my daughter Lesira for their encouragement.

Maputo, June 25, 1996

Paulus Gerdes

PREFACE

(Second edition)

The new edition of the book *Lunda Geometry* contains two additional chapters. Chapter 9, entitled *On the Geometry of Celtic knots and their Lunda-designs*, was published in the British journal *Mathematics in School* (1999). Chapter 10 is the last section of an invited paper presented at the Wenner-Gren International Symposium "Symmetry 2000" (Stockholm, September 13-16, 2000) and published in the proceedings. The new edition does not include the review of Solomon Golomb's book *Polyominoes*, published in *Archives Internationales d'Histoire des Sciences* (1998, 48, No. 140, 174-176).

Chapter 4 of my book *Geometry from Africa: Mathematical and Educational Explorations*, published by the MAA (1999), presents an introduction to Lunda-designs and includes a section on Lunda-polyhedral-designs and a board game. The relationship between Lunda-designs and magic squares is explored in a paper published in *The College Mathematics Journal* (2000).

The study of Lunda-designs led to the discovery of Liki-designs (2002a, 2002b) and of various types of matrices, including cycle matrices (see e.g. the papers published in the electronic journal *Visual Mathematics*). The book *Adventures in the World of Matrices* (2007d) presents an introduction to cycle matrices. Further books on the beautiful geometry and linear algebra of Lunda-designs are forthcoming.

The paper *Lunda Symmetry where Geometry meets Art* (2005) explores some relationships between Lunda Geometry and art. Mathematician and artist John Sims of the Ringling School of Art and Design (Sarasota, Florida) organized a math-art exhibition including some of colorful Lunda-designs I had prepared. A book on Lunda-designs and art is in preparation.

Maputo, October 2007
Paulus Gerdes

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Chapter 1

ON *SONA* SAND DRAWINGS, MIRROR CURVES AND THE GENERATION OF LUNDA-DESIGNS

1.1 About *sona* sand drawings

The *sona* tradition belongs to the heritage of the Cokwe and neighboring peoples in eastern Angola, and northwestern Zambia. When the Cokwe gathered at their village meeting places or at their hunting camps, they usually sat around a fire or in the shadow of leafy trees spending their time in conversations illustrated by drawings in the sand. These drawings are called *lusona* (singular) or *sona* (plural).

Each boy learnt the meaning and execution of the easier *sona* during their period of intensive schooling, the initiation rites. The more complicated *sona* were transmitted by specialists, called the *akwa kuta sona* (those who know how to draw), to their male descendants. These drawing experts were at the same time the storytellers who used the sand drawings as illustrations for proverbs, fables, games, riddles and animals.

In order to facilitate the memorization of their standardized *sona*, the *akwa kuta sona* invented an interesting mnemonic device. After cleaning and smoothing the ground, they first set out an orthogonal grid of equidistant points with their fingertips. The number of rows and columns depends on the motif to be represented. Then they draw a line figure that embraces all the grid points. To do so they apply the geometrical algorithm that corresponds to the motif to be represented. Figure 1.1 displays an example: the line figure represents the path followed by a wild chicken trying to escape its hunters.

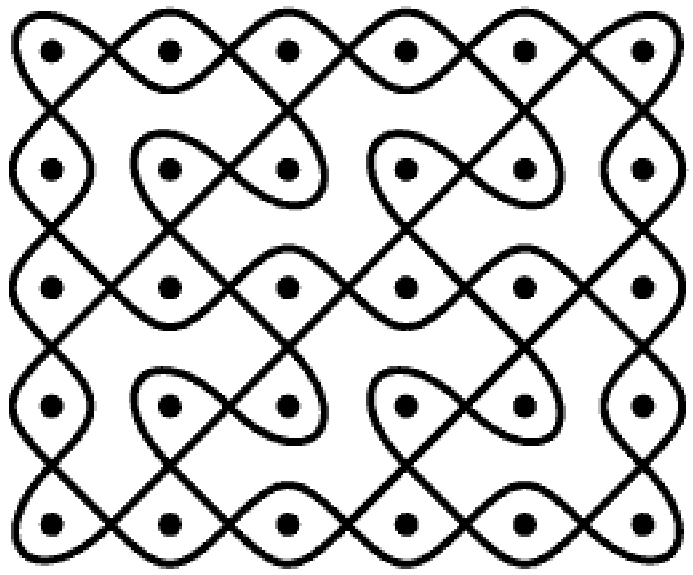

Figure 1.1

An analysis of the *sona* sand drawing tradition and a contribution to its reconstruction is presented in the first volume of my book *Sona Geometry — Reflections on the tradition of sand drawings in Africa South of the equator* (1994, 2006).

1.2 Towards a discovery [1]

When one studies a proof one rarely learns how the mathematician discovered his result. The path that leads towards a discovery is generally very different from the paved road of the deduction. The path to the discovery begins by zigzagging across a densely vegetated area full of obstacles, and apparently without exit, until suddenly it comes to an open space with flashes of surprise. Almost immediately the delight of the unexpected "heureka" (Greek: "I found", "I discovered") opens the road triumphantly.

Often confronted with students' question about how I discovered the theorems which will be proved in the next section, I will now try to re-open the road in the hope of stimulating mathematical research by new generations of *akwa kuta sona* — drawing experts. Once the road is reconstructed, the mystery of inspiration is solved.

In order to facilitate the execution of the *sona* sand drawings I was analyzing, I became used to drawing them on squared paper with a distance of two units between two successive grid points (see Figure 1.2).

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1 This section is a partial translation of chapter 6 of the author's book *Geometria Sona*, Vol. 2, ISP, Maputo, 1993. The questions to the readers and the problem section have been deleted.

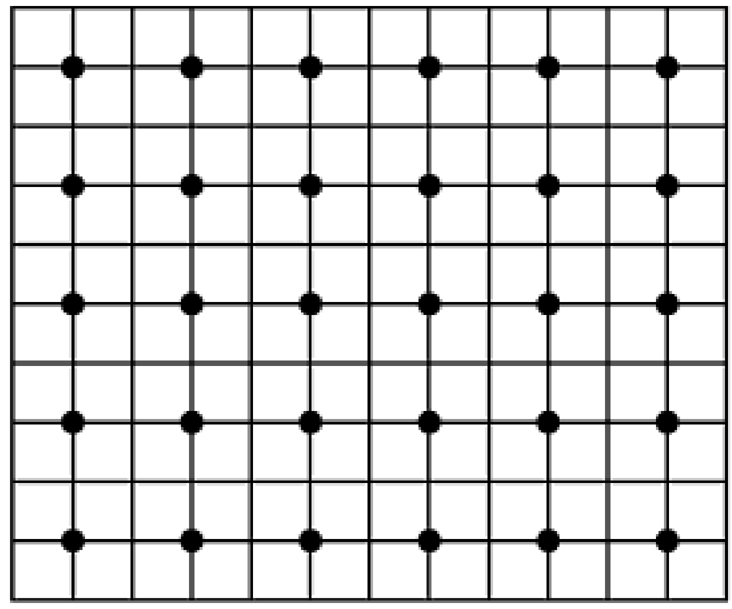


Figure 1.2

In this way, a monolinear drawing like the "chased chicken path" (see Figure 1.3) passes exactly once through each of the small squares inside the circumscribed rectangle.

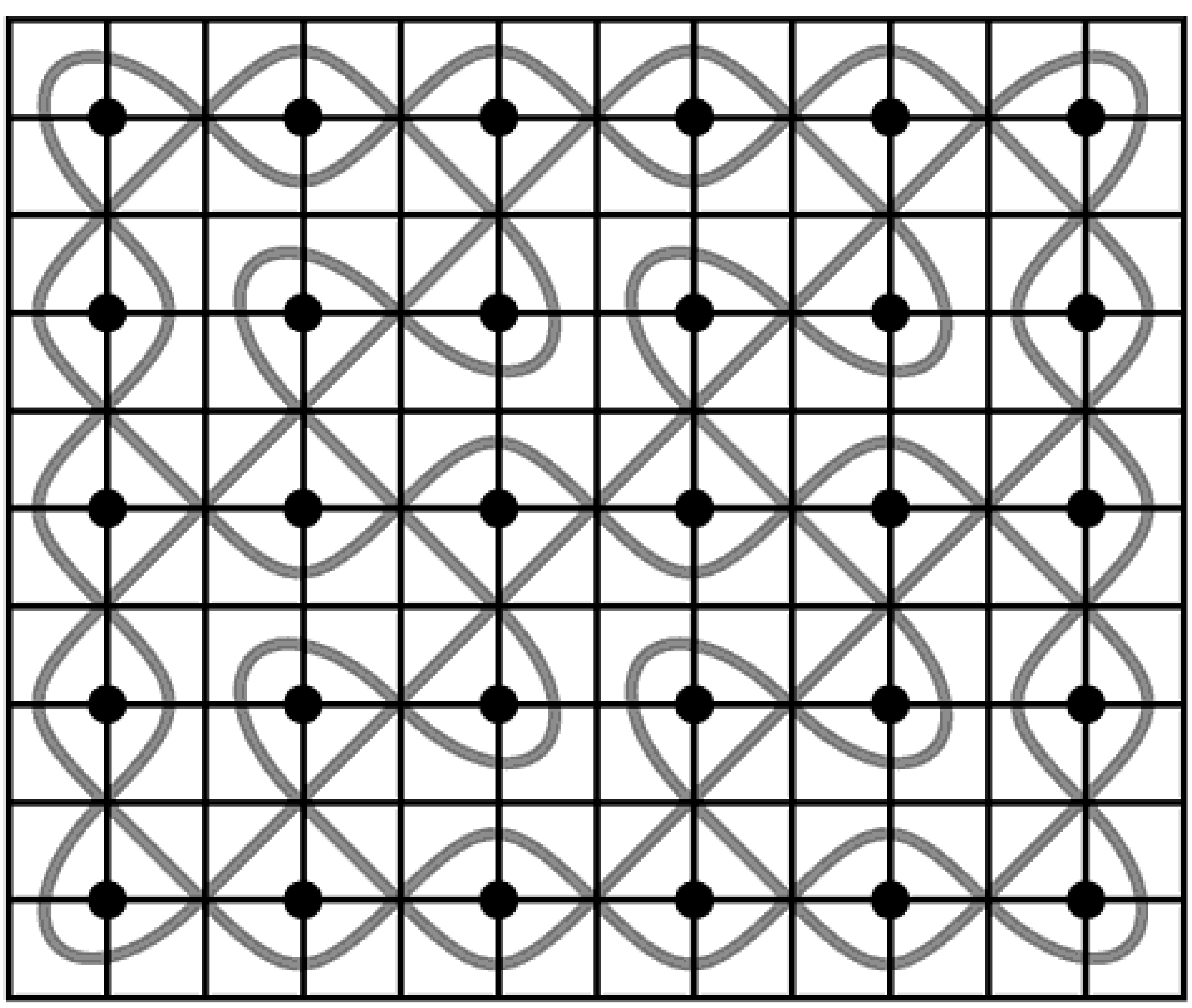


Figure 1.3

This allows the possibility of **enumerating** the small squares, **1** being the number attributed to the small square where one starts the line, and **2** the number of the second unit square through which the curve passes, and so on successively until the closed curve is complete. See the example begun in Figure 1.4 and concluded in Figure 1.5.

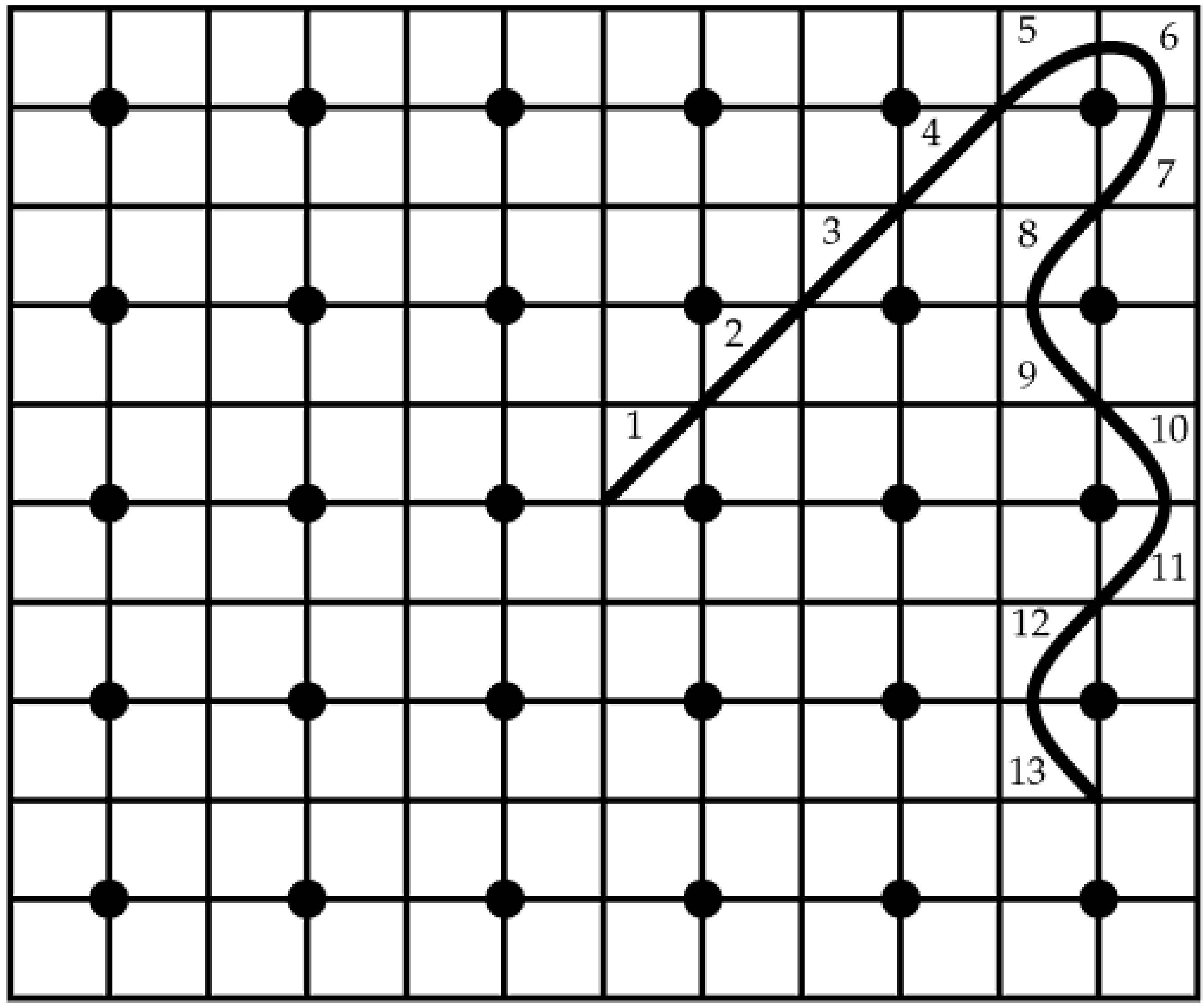


Figure 1.4

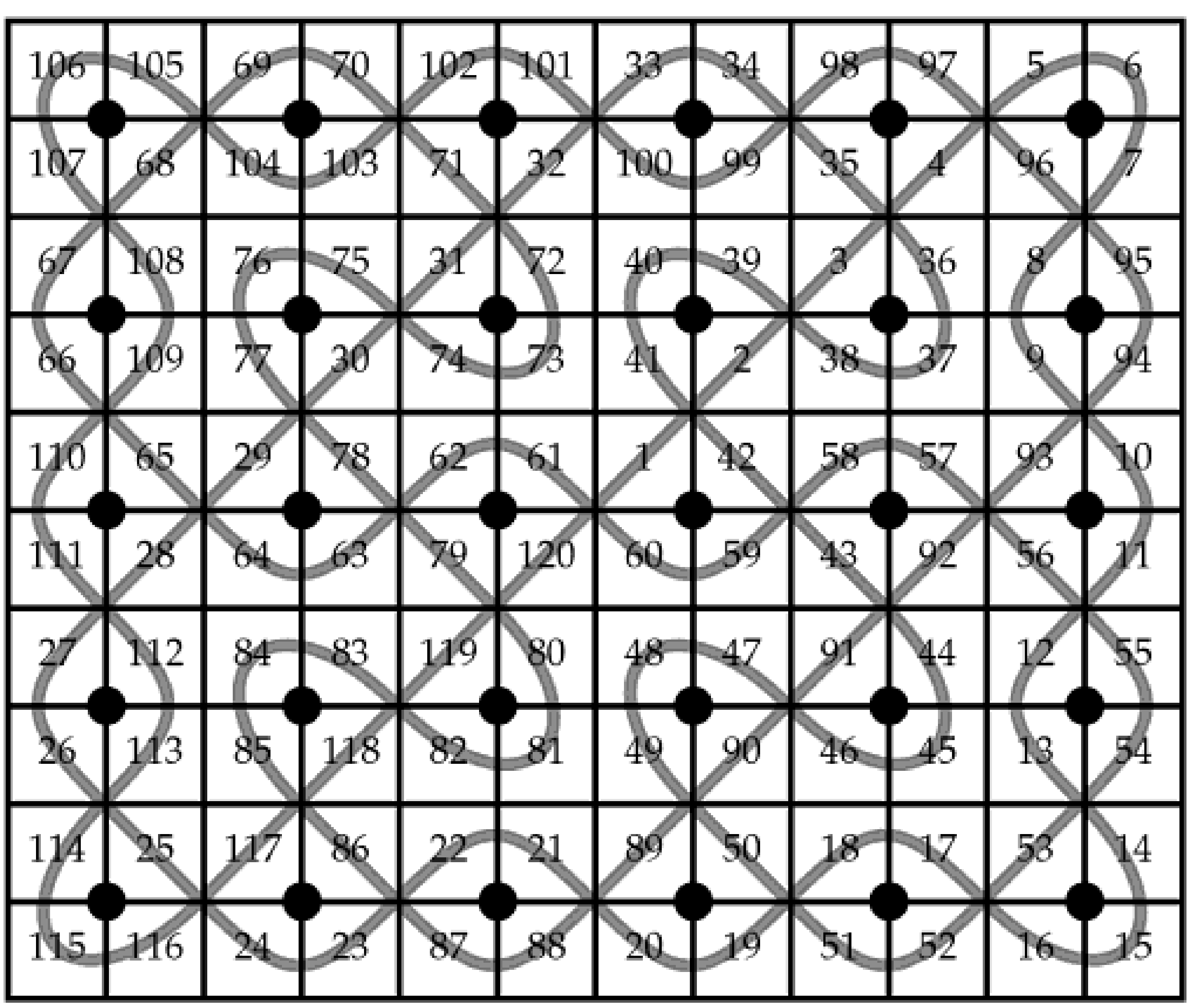


Figure 1.5

The path followed by the "chased chicken" is aesthetically attractive. The design displays a rotational symmetry of 180^{o} (see

Figure 1.1 once more). This leads to the following question: How is the beauty and the symmetry of this sand drawing reflected in the enumeration of the small squares?

For example, what relationship does exist between two small squares, which correspond to each other under a rotation of 180^{o}? The first number of the first row, 106, corresponds to the last number of the last row, 15; the second number of the first row, 105, corresponds to the penultimate number of the last row, 16. In both cases, the sum of the numbers of the two corresponding small squares is equal to 121. Will the same happen in the other cases? The small square with number 72 corresponds to the small square with number 49; the small square of number 93 corresponds to the small square with number 28, etc. (see Figure 1.6). The sum is always equal to 121, that is, equal to the number of the last small square in the enumeration, plus one.

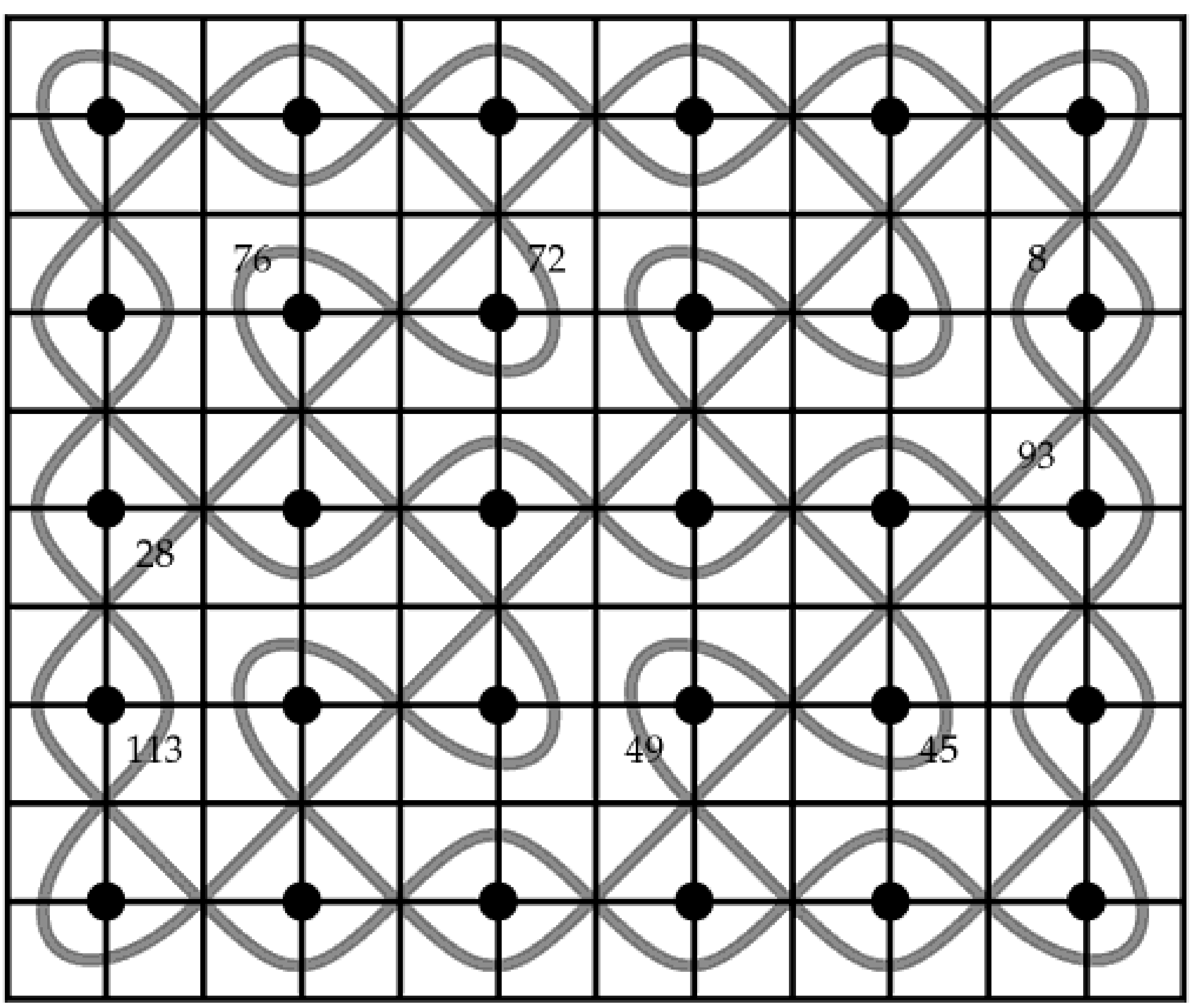


Figure 1.6

The reader is invited to find a proof for the truth of this affirmation. What will happen if we start the enumeration in another small square or in another direction: Will the sum of the numbers of two small squares, which correspond under a rotation of 180^{o}, always continue to be equal to 121?

Will the beauty of the sand drawing under consideration also be reflected in other ways in the enumeration of the small squares?

When we enumerate the small squares, we obtain a rectangle of numbers. Will this numerical rectangle be interesting, that is, for example, '**magic**'? A numerical rectangle is called 'magic' if, for all rows, the sums of the numbers of their small squares are equal and if, at the same time, for all columns, the sums of the numbers of their small squares are equal too. Figure 1.7 displays the sums of the numbers row by row. Only some of them are equal. Have we come to a dead-end?

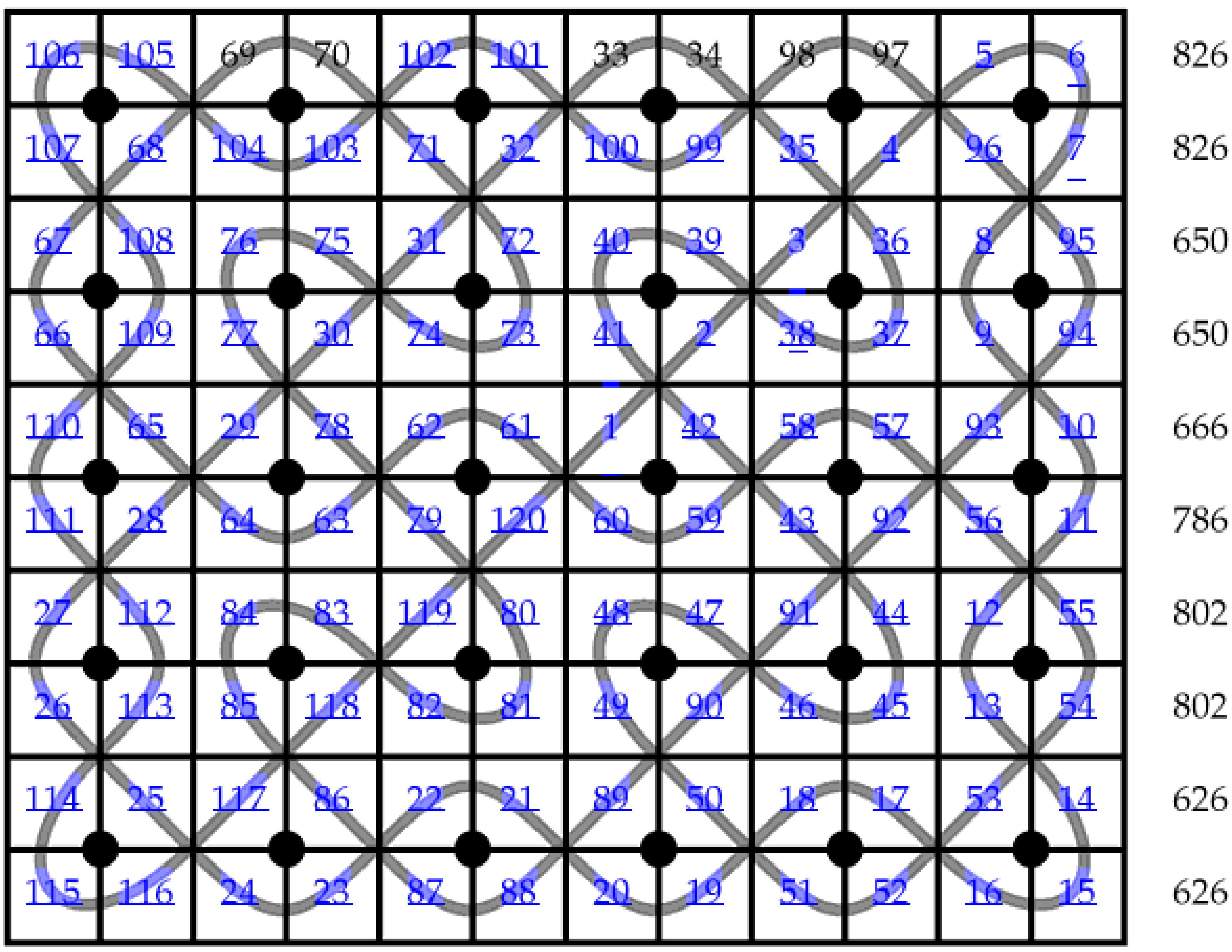


Figure 1.7

Let us consider a similar, smaller design (see Figure 1.8) and let us count the smaller squares from the center outwards. Figure 1.9 shows the result. Calculating the sums of the numbers, row by row, and, column by column (see Figure 1.10), we verify that the sums of four rows are equal to 196.

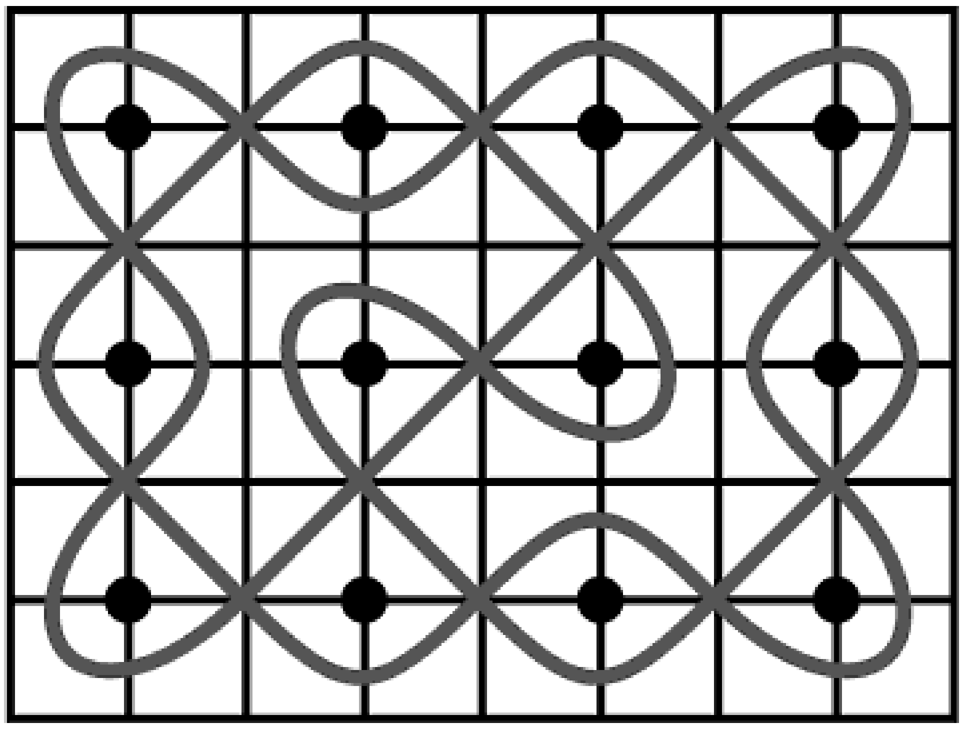


Figure 1.8

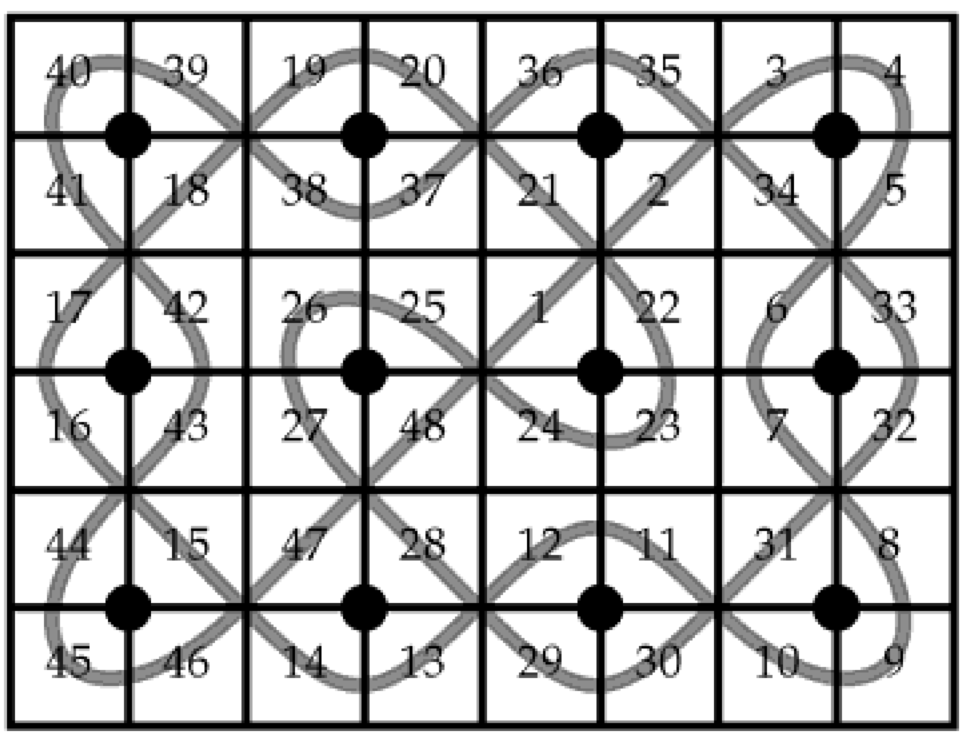


Figure 1.9

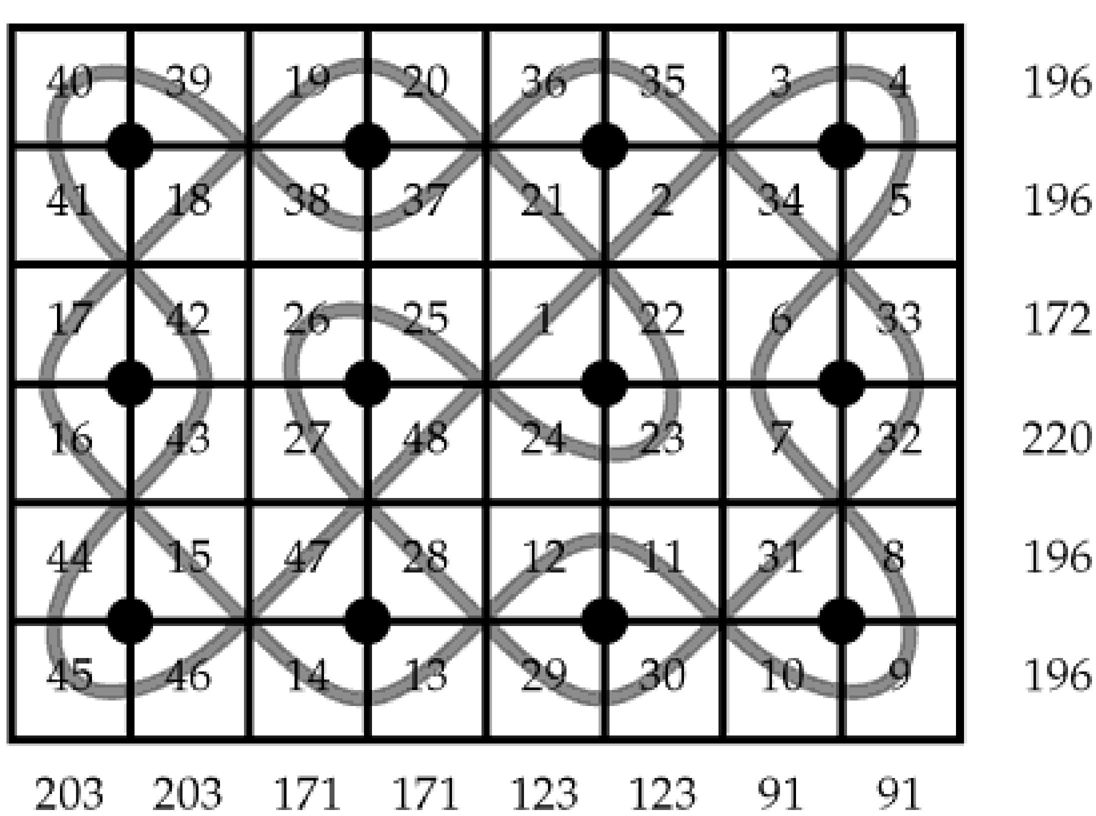


Figure 1.10

We would like to see that the six sums are equal, but only four are. Bad luck... The numerical rectangle is not ‘magic’..., or could it sometimes be that

$$220 = 196 = 172 \ ?$$

Distinct numbers never can be really equal; at most they may be equivalent or equal modulo m. This means that their difference is a multiple of the natural number m.

For which values of $\boldsymbol{m}$ may $220 = 196 = 172$ modulo m happen?

If $220 = 196$ modulo m, then their difference 220-196, that is 24, has to be a multiple of m.

We would also like to see that the sums of the numbers in the columns are equal:

$$203 = 171 = 123 = 91.$$

As they are in fact not equal, we would prefer that they are equal modulo the same number $\boldsymbol{m}$. Therefore, 203-171, that is 32, has to be a multiple of m. As both 32 and 24 are multiples of m, 32-24, that is, 8 also has to be a multiple of m. In this way we see that $\boldsymbol{m}$ may **only** be 8, 4 or 2. Let us analyze the possibility m=8.

Instead of counting naturally the small squares through which the line passes, that is, 1, 2, 3, 4, 5, ..., 48, let us enumerate them modulo 8:

1, 2, 3, 4, 5, 6, 7, 0, 1, 2, 3, 4, 5, 6, 7, 0, ...

Figure 1.11 shows the start of the enumeration modulo 8 and Figure 1.12 the final result. We note that the numerical rectangle that is thus obtained is ‘magic’ modulo 8, as 28=20=36=4 modulo 8 and 11=27=19=3 modulo 8.

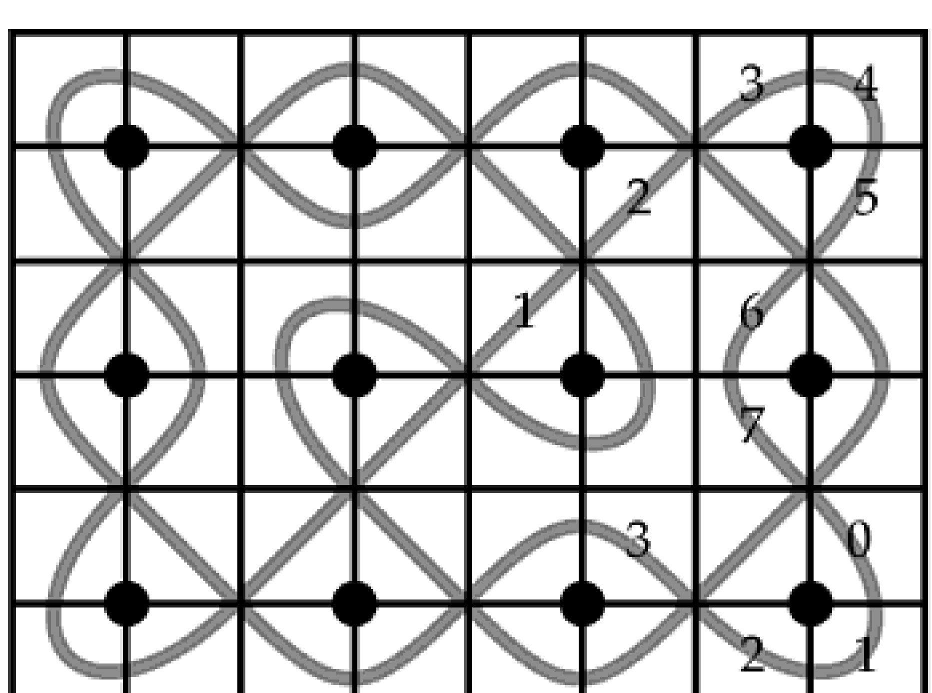


Figure 1.11

Let us now attentively observe the distribution of the numbers 1, 2, 3, 4, 5, 6, 7, 0, throughout the rectangle. What happens to the numbers of four small squares, which touch the same grid point?

We may see that, in most cases, four consecutive numbers appear around a grid point:

* 3, 4, 5, 6 around the second point of the first row of the grid;
* 2, 3, 4, 5 around the third point of the first row of the grid, etc.

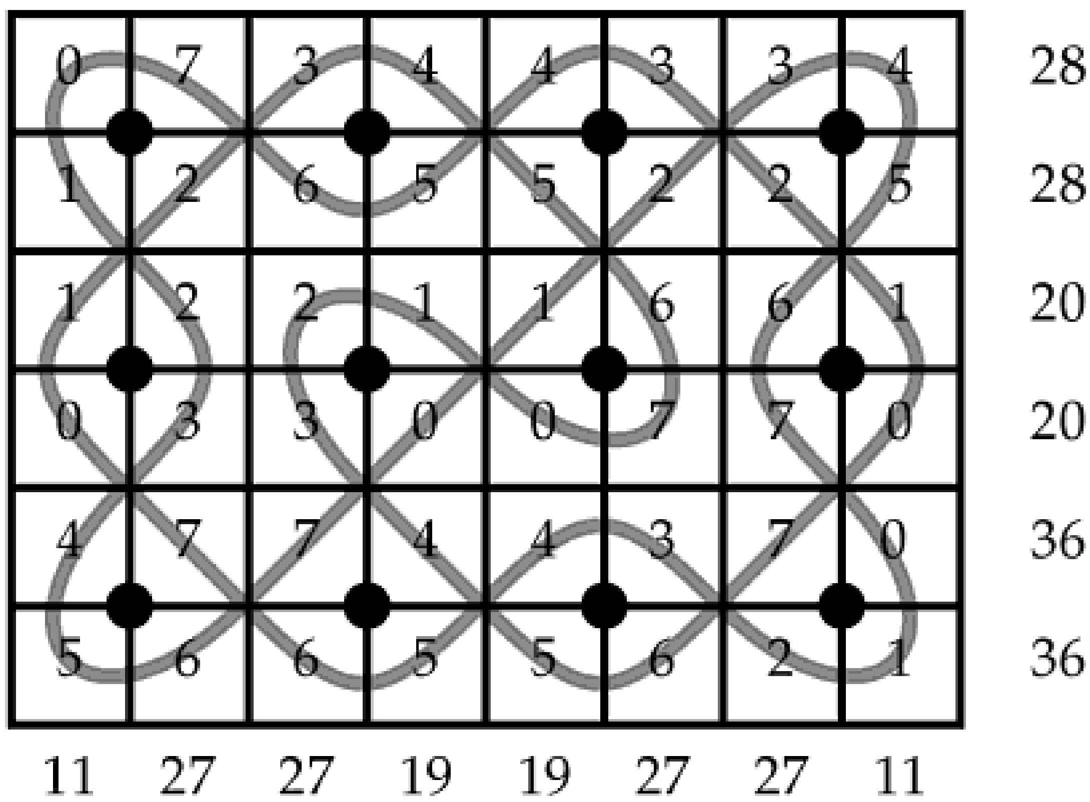


Figure 1.12

In only four cases does this not happen. For example, around the first point of the first row, we find 0, 1, 2, 7 instead of 0, 1, 2, 3; around the third point (on the left hand side) of the second row, we find 0, 1, 6, 7 instead of 0, 1, 2, 3. What is to be done?

If only 6=2 and 7=3, then the situation would be ‘normalized’. Counting modulo 4 or modulo 2, we have 6=2 and 7=3.

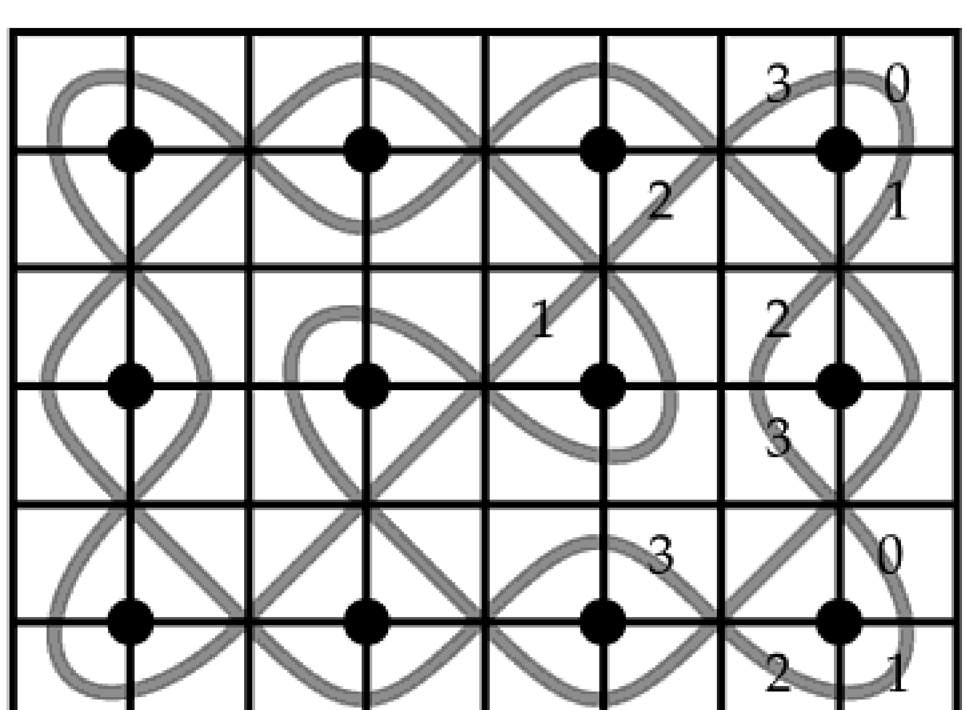


Figure 1.13

Let us now enumerate modulo 4 instead of modulo 8 the small squares through which the curve passes successively. Figure 1.13 shows the beginning of the enumeration modulo 4 and Figure 1.14 the conclusion:

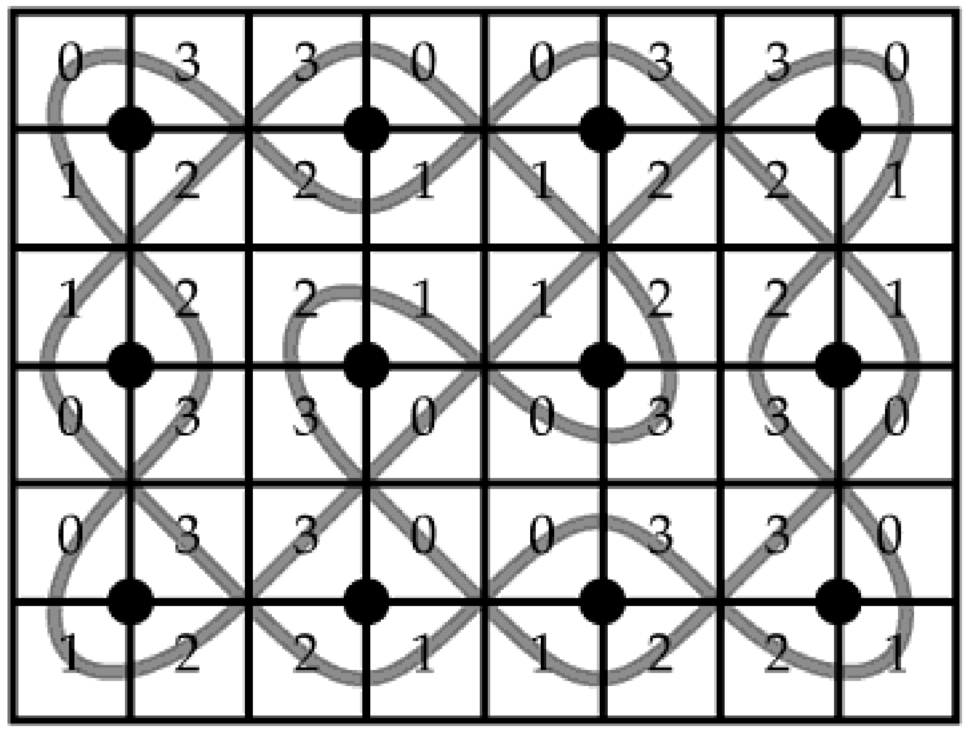


Figure 1.14

Now we find the numbers 0, 1, 2, and 3 around all grid points; the rectangle of the small squares remains ‘magic’. Moreover, we have won new and beautiful surprises: the disposition of 0, 1, 2, 3 is alternately clockwise and anti-clockwise (see Figure 1.15); between four neighboring grid points there are always four equal numbers (see Figure 1.14 once again).

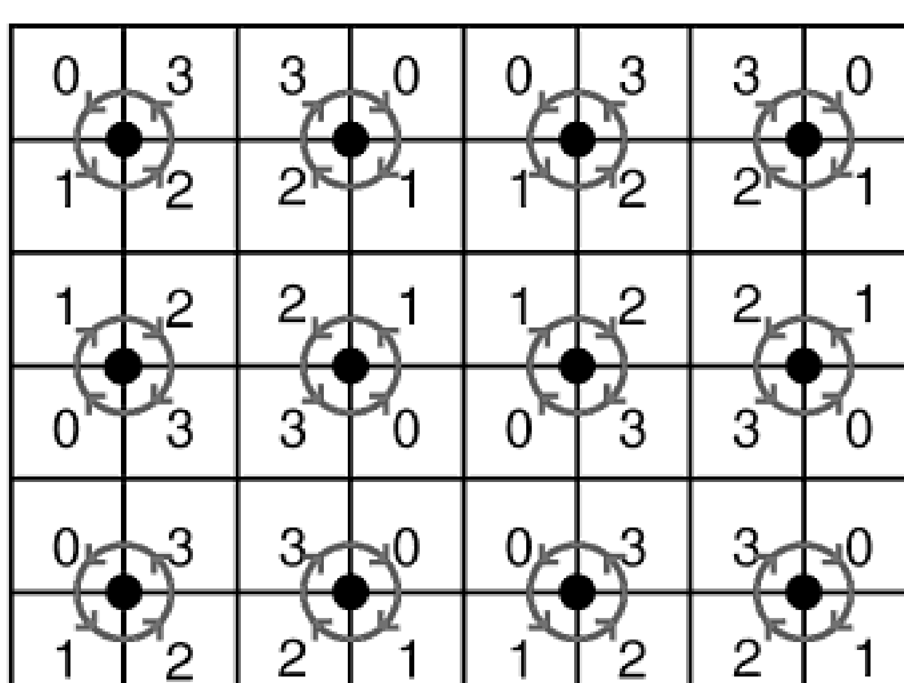


Figure 1.15

Will the same happen with the larger ‘chased chicken’ sand drawing in Figure 1.3, and with other regular and monolinear sand drawings like that of the ‘lion’s stomach’ (see Figure 1.16)?

For which drawings the same phenomenon is verified remains to be seen. The reader is invited to experiment and to find a general answer.

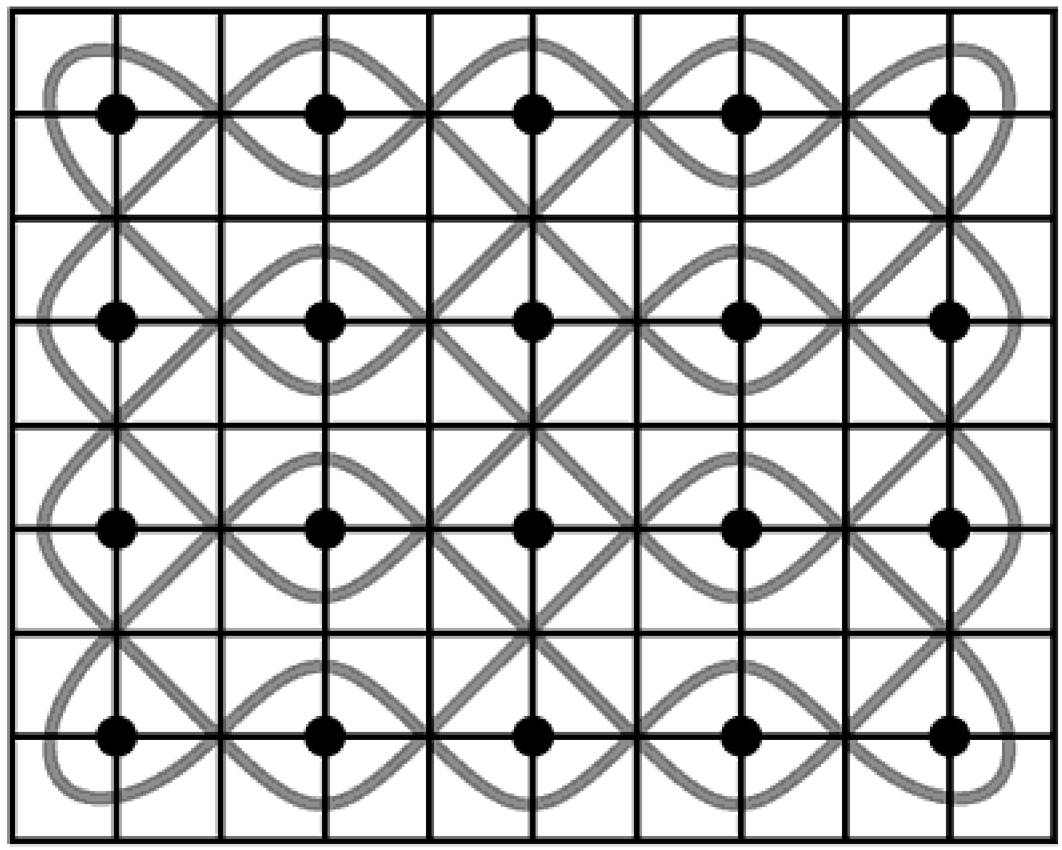


Figure 1.16

In the following section a possible answer will be presented.

1.3 Some theorems about smooth and monolinear mirror line designs

Introduction

A whole class of drawings to which the ‘chased chicken’ and ‘lion’s stomach’ designs belong, which we met in the previous section, satisfies a common construction principle. The curves involved may be generated in the following way: each of them is the smooth version of a closed polygonal path described by a light ray emitted from the point A (0,1) (see Figure 1.17a). The light ray is reflected on the sides of the circumscribed rectangle of the grid, and on its way through the grid it encounters double-sided mirrors. These mirrors are placed vertically in the center between two horizontal neighboring grid points and horizontally in the center between two vertical neighboring grid points. Figure 1.17 shows the generation of the ‘chased chicken’ drawing.

In the following we will define the designs that satisfy the aforementioned construction principle and demonstrate a few theorems which reveal some properties of this class of designs.

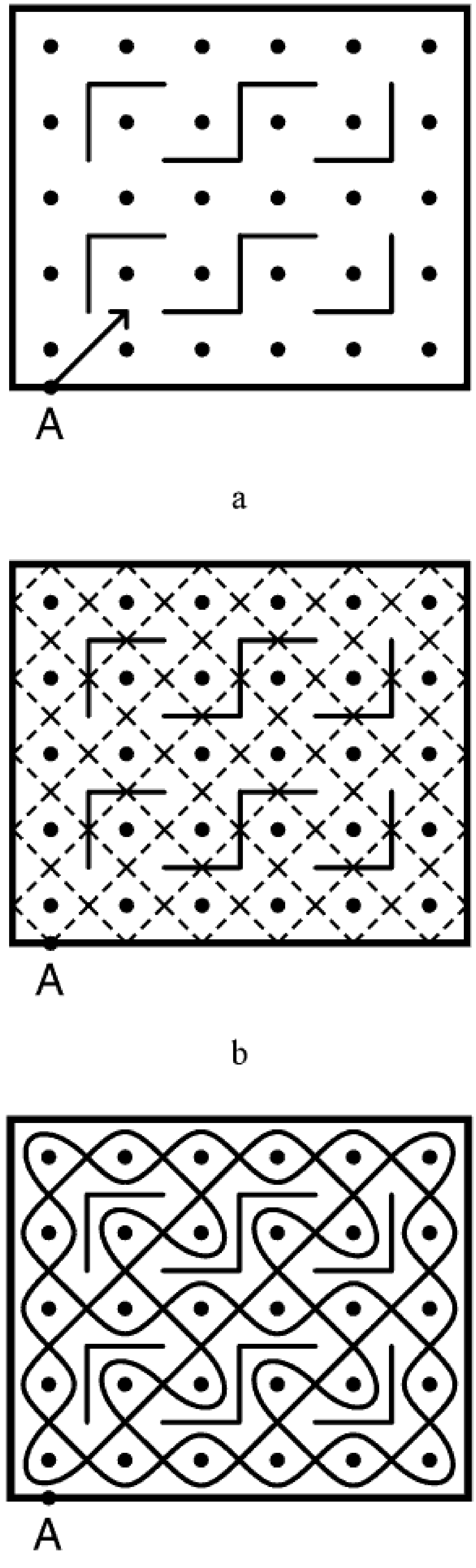
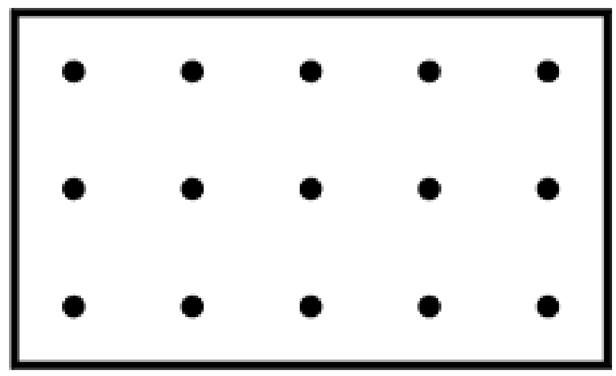


Figure 1.17

Consider a **rectangular grid RG[m,n]** with vertices (0,0), (2m,0), (2m,2n) and (0,2n) and having as points (2s-1,2t-1), where s = 1,..., m and t = 1,..., n, and ***m*** and ***n*** two arbitrary natural numbers.

Figure 1.18 displays the example RG[5,3].



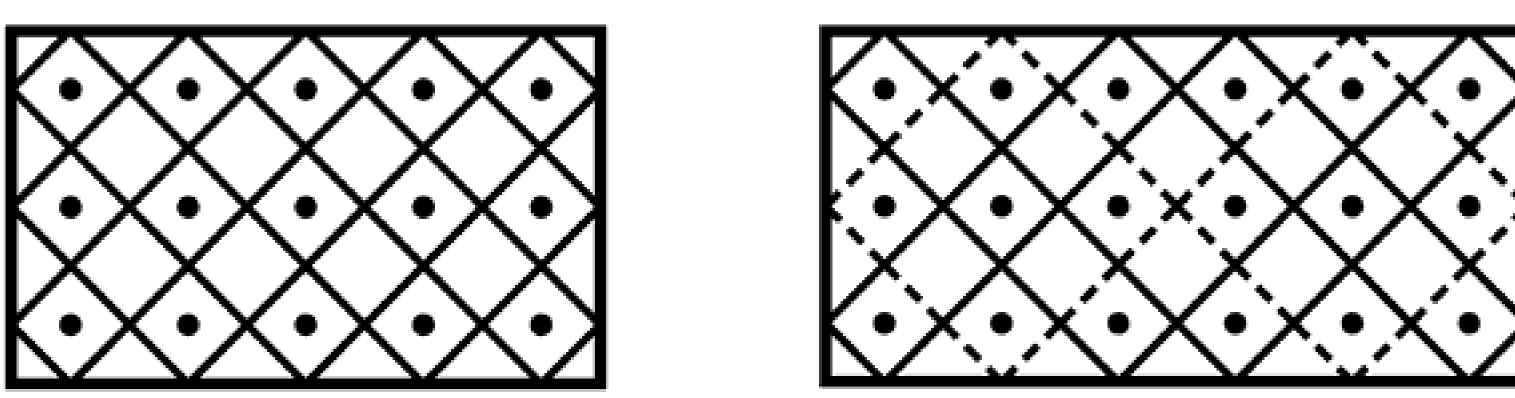
RG[5,3]
Figure 1.18

The intersection of RG[m,n] with the set of straight lines

$$y = \pm x + (2u+1),$$

where ***u*** represents an arbitrary whole number, will be called a **diagonal design D[m, n]**.

Figure 1.19 shows the examples D[5,3] and D[6,3].



a b

Figure 1.19

A diagonal design may be considered as the union of the "polygonal mirror lines" which are traced by light rays emitted from the points (2s-1,0) in the direction of (2s,1), and which are reflected on the sides of the rectangle (s = 1, 2, ..., m).

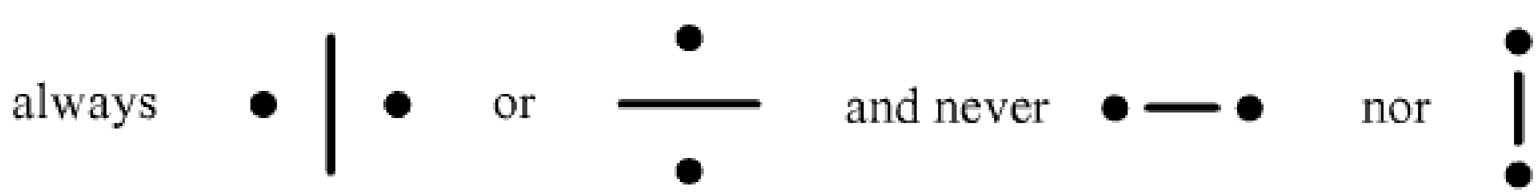
We call a diagonal design **p-linear**, if it is composed of *p* distinct closed "polygonal mirror lines".

For example, D[6,3] is 3-linear and D[5,3] is monolinear (1-linear).

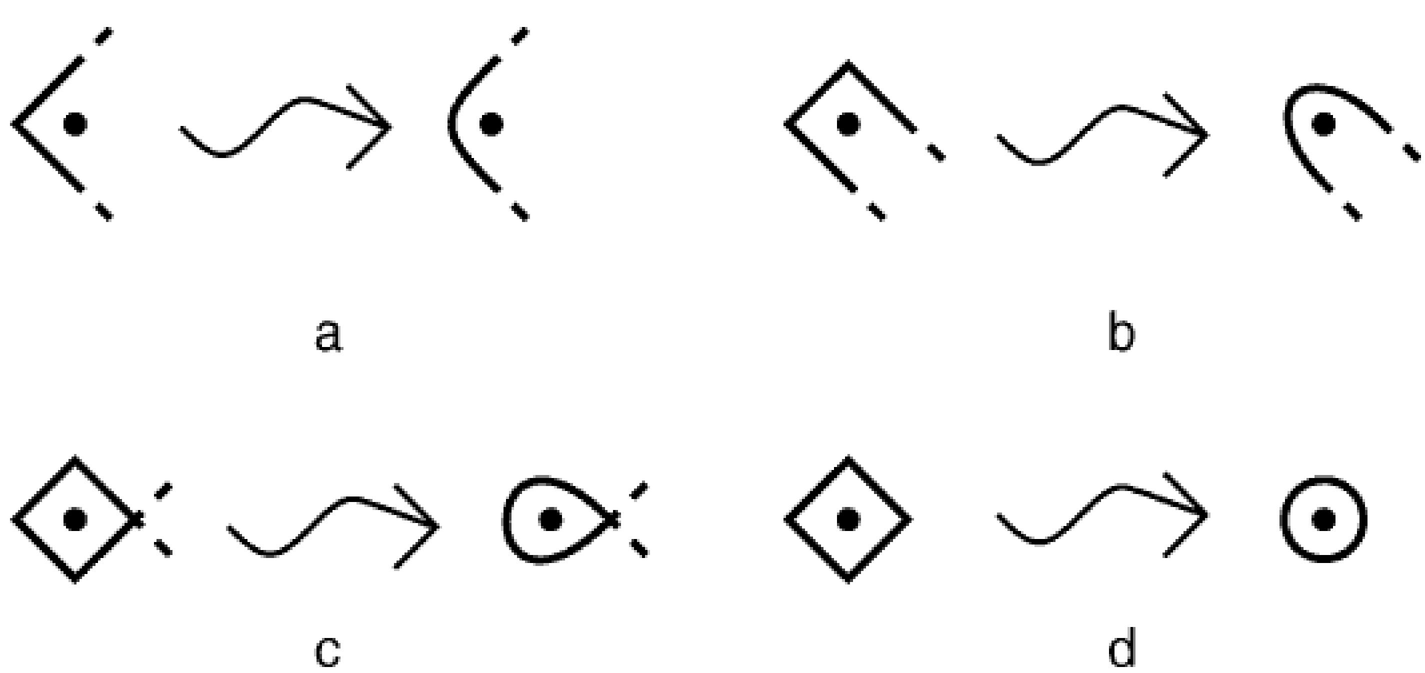
When horizontal and vertical double-sided mirrors of unit length are placed in a diagonal design in the midpoints between horizontal and vertical neighboring grid points, we call it a **polygonal mirror lines design**.

A polygonal mirror lines design may be considered as the union of the polygonal paths described by light rays emitted from the points (2s-1,2t) in the direction of (2s,2t+1), and which are reflected on the mirrors and the sides of the rectangle (s = 1,..., m; t = 1,..., n-1). We call a polygonal mirror lines design **p-linear**, if it consists of ***p*** distinct closed polygonal paths.

A polygonal mirror lines design will be called **regular** when all mirrors between horizontal neighboring points are always in the vertical position and when, at the same time, all mirrors between vertical neighboring points are always in the horizontal position:

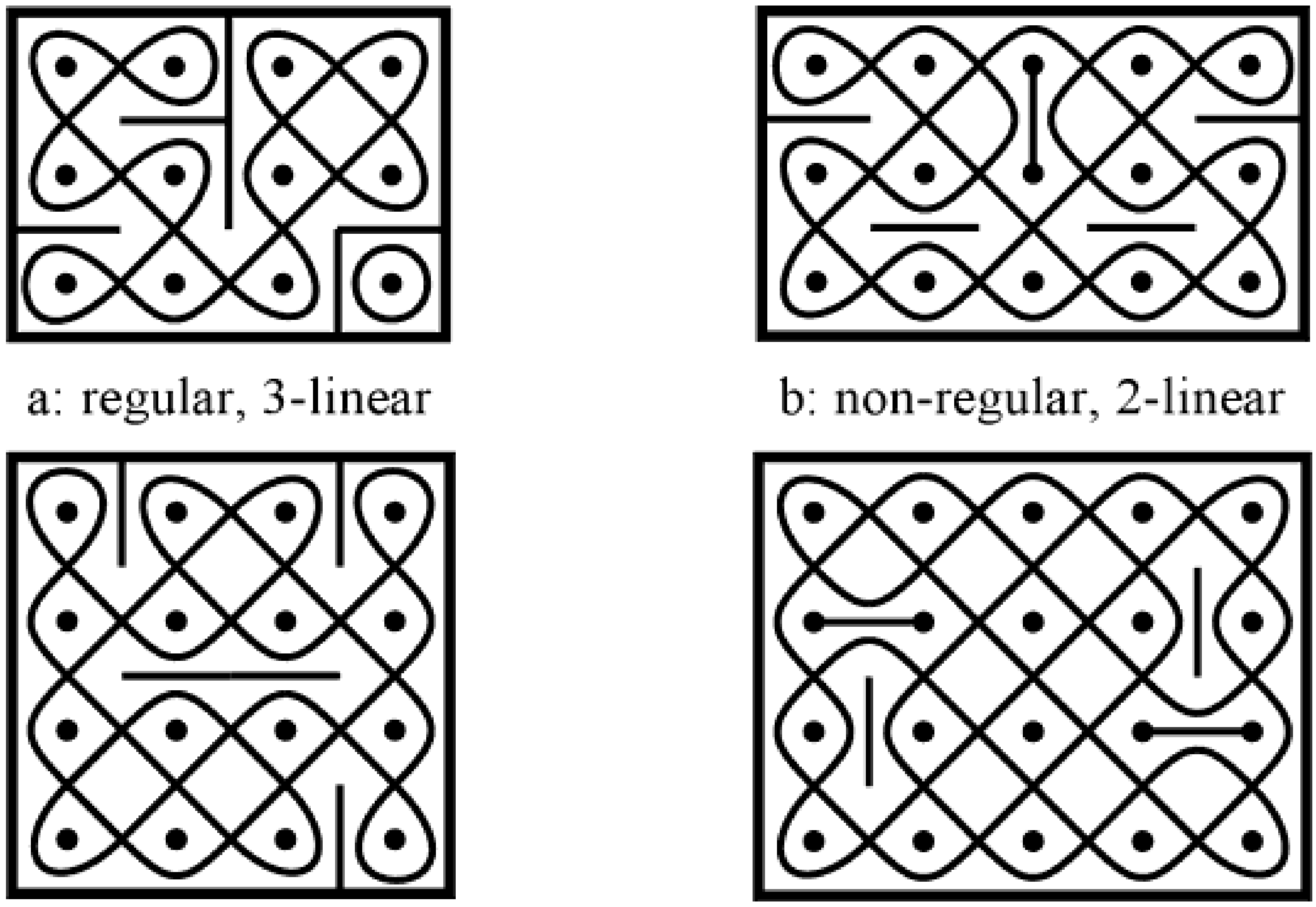


When all polygonal elements of a polygonal mirror lines design are transformed into smooth curve elements, in agreement with the transformation rules represented in Figure 1.20, we will call the result a **smooth mirror lines design**.



Transformation rules
Figure 1.20

Inversely, we may consider a polygonal mirror lines design as the 'rectification' of a smooth mirror lines design. Figure 1.21 presents examples.



a: regular, 3-linear

b: non-regular, 2-linear

c: regular, monolinear

d: non-regular, monolinear

Figure 1.21

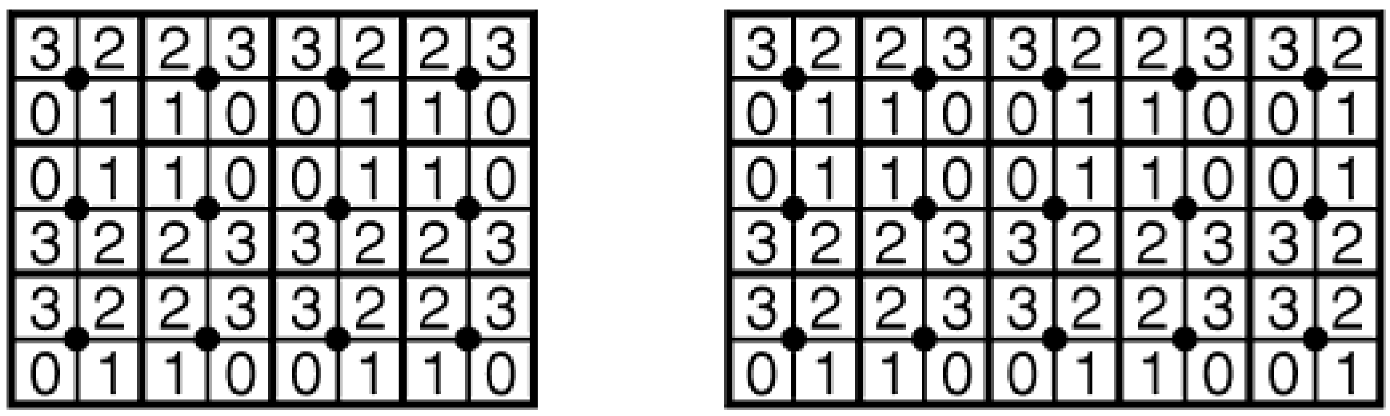
Consider now the unit squares of the initial rectangular grid RG[m,n], that is, the squares whose vertices have the coordinates (p,q), (p+1,q), (p+1,q+1) and (p,q+1), where p = 0,1,..., 2m-1 and q = 0,1,..., 2n-1. Each of these unit squares has one unique grid point as one of its four vertices. The unit squares may be enumerated, as Figure 1.22 illustrates, in dependence of their position relative to the corresponding grid point (2s-1,2t-1). This enumeration will be called **Q-enumeration (modulo 4)**.

		s	
		odd	even
t	odd	3 2 • 0 1	2 3 • 1 0
	even	0 1 • 3 2	1 0 • 2 3

Q-enumeration (modulo 4)

Figure 1.22

Figure 1.23 shows the Q-enumeration of the rectangular grids R[4,3] and R[5,3]. We note that the same number is attributed — as a consequence of the definition — to the four unit squares, which belong to the same square of neighboring grid points.



Q-enumeration of RG[4,3] and RG[5,3]
Figure 1.23

Consider a monolinear, smooth mirror lines design, or, in brief, a (rectangle-filling) mirror curve. Let us assume that the closed curve is gone through in the following way: one starts in the unit square $[A_0]$ with vertices (1,0), (2,0), (0,1) and (1,1). Let $\mathbf{A_g}$ be the g^{th} attained unit square through which the curve goes. As the mirror lines design is monolinear, the curve passes through all unit squares of the rectangular grid. This makes it possible to introduce a second enumeration of the unit squares: the **p-number** of the unit square $\mathbf{A_g}$ is defined by **g modulo 4**, that is

$$P(A_g) = g \mod 4.$$

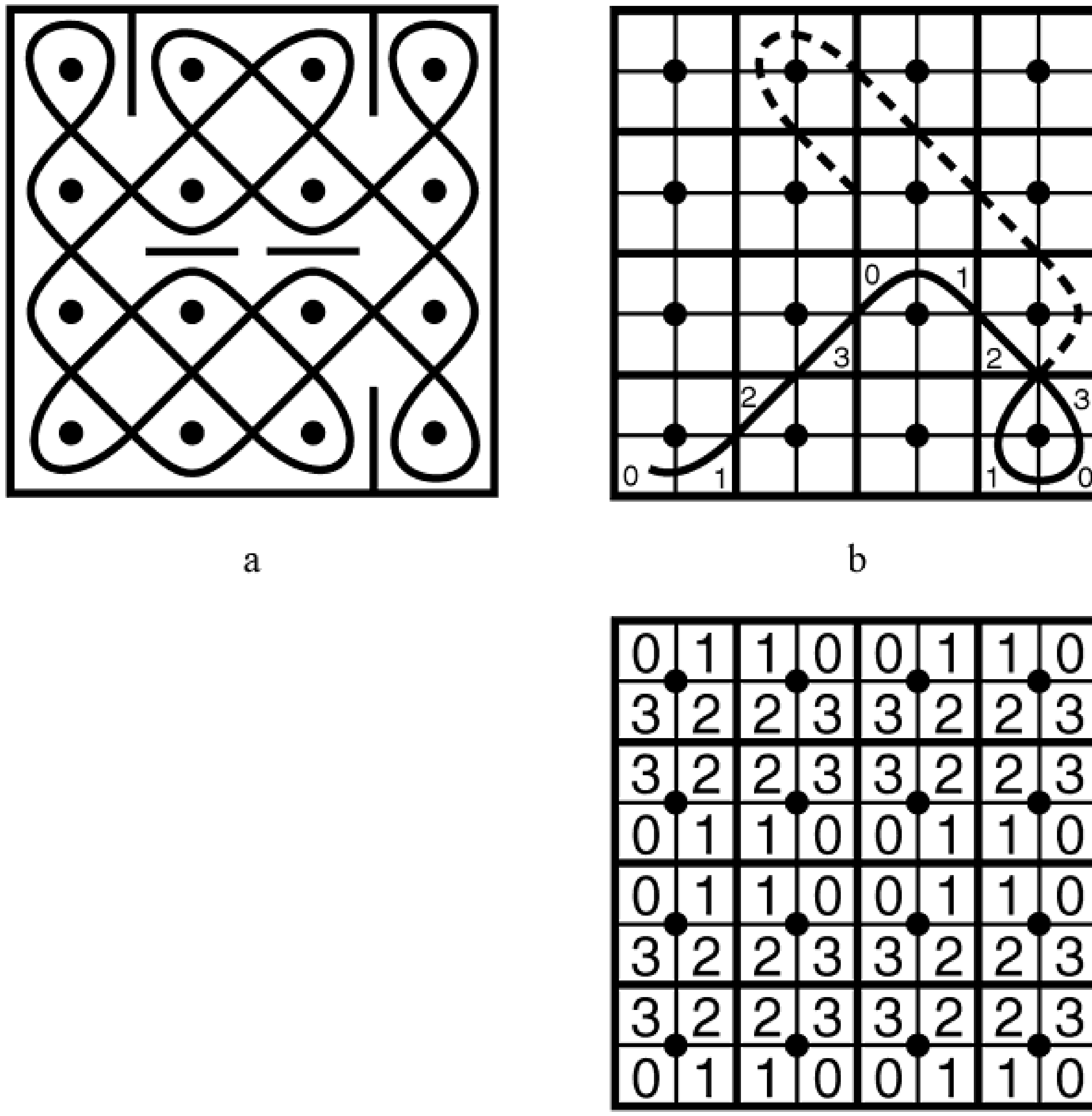
Figure 1.24 gives an example.

Now we will demonstrate the (surprising) theorem that states that — in the case of monolinear, regular and smooth mirror lines designs — the two enumerations are equal, that is

$$Q(A_g) = P(A_g) \text{ for } g = 0,1,\ldots,4mn\text{-}1.$$

To facilitate the demonstration, we first prove the following auxiliary theorem:

Theorem 1: For monolinear, regular and smooth mirror lines designs, the following equality $Q(A_{g+2}) = Q(A_g) + 2 \pmod 4$, holds for $g = 0,1,\ldots, 4mn\text{-}1$.



a b c

Example of a P-enumeration

Figure 1.24

Proof:

Consider three unit squares through which the curve successively passes. When going through the three unit squares, the curve may encounter 0, 1, 2 or 3 mirrors. In this way, we may distinguish five essentially different situations (see Figure 1.25). In each situation we have that the first (A_g) and the third (A_{g+2}) unit square, through which the curve passes, belong to diagonally opposed grid point squares (see Figure 1.26).

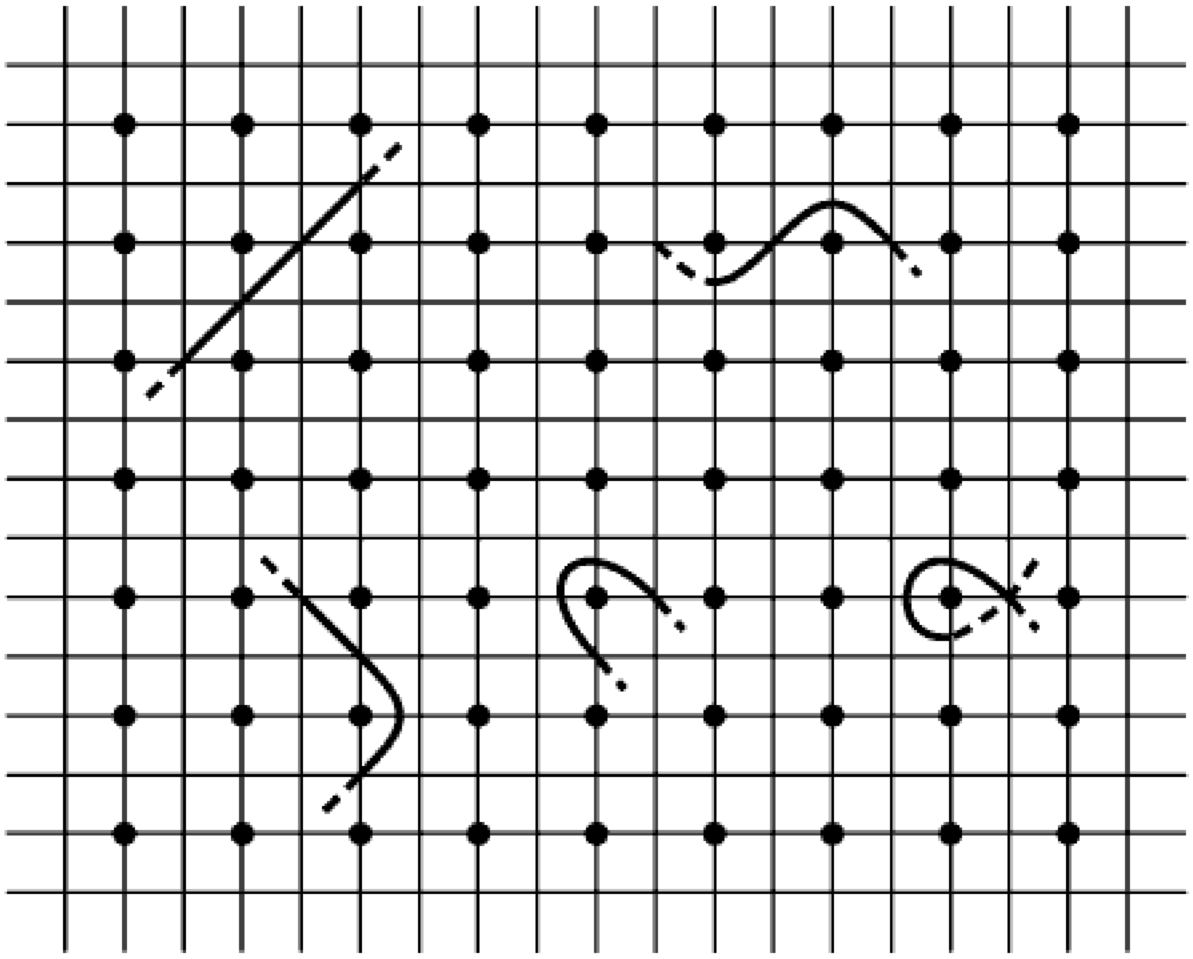


Figure 1.25

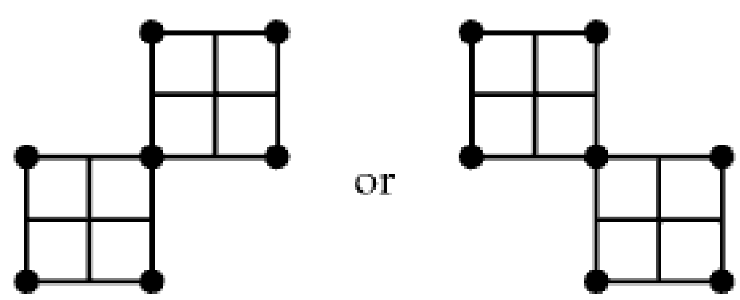


Figure 1.26

In agreement with the definition of the Q-enumeration, it follows immediately that

$$Q(A_{g+2}) = Q(A_g) \ +2 \ \ (\text{mod } 4),$$

as we wished to prove.

Theorem 2: For monolinear, regular and smooth mirror lines designs the following equality $Q(A_g) = P(A_g)$ holds for $g = 0, 1, \ldots, 4mn\text{-}1$.

Proof:

In accordance with the definitions of the P-enumeration and of the Q-enumeration we have:

(1) $P(A_0) = 0 = Q(A_0)$ and

(2) $P(A_1) = 1 = Q(A_1)$.

For $g = 2, 3, ..., 4mn-1$, we have

$$P(A_g) = g \pmod 4 \text{ and}$$
$$P(A_{g+2}) = g+2 \pmod 4.$$

Therefore,

(3) $$P(A_{g+2}) = P(A_g) + 2 \pmod 4.$$

In agreement with (1), (2) and (3) and theorem 1, it follows that

$$Q(A_g) = P(A_g) \text{ for } g = 0, 1,..., 4mn-1,$$

as we wished to prove.

Corollary 1: Two neighboring parallel line segments of a monolinear, regular and smooth mirror lines design are always traversed in opposite directions.

Proof:

If the straight line segment I is traversed in the direction a*b , where a*b = 0*1, 1*2, 2*3 or 3*0 (mod 4), the line segment II is also traversed in the direction a*b, that is, in the opposite direction (see Figure 1.27a).

If the curved line segment III (see Figure 1.27b) is traversed in the direction a*c (upwards), this implies, by consequence of the definition of the Q-enumeration, that a=3 and c=0 or a=1 and c=2. In the first case we obtain b=2 and d=1, that is, the curved line segment IV is traversed in the direction d*b (downwards). In the second case b=0 and d=3 hold and the curved line segment IV is traversed in the direction d*b (downwards). In other words, in both cases the neighboring curved line segments are traversed in opposite directions.

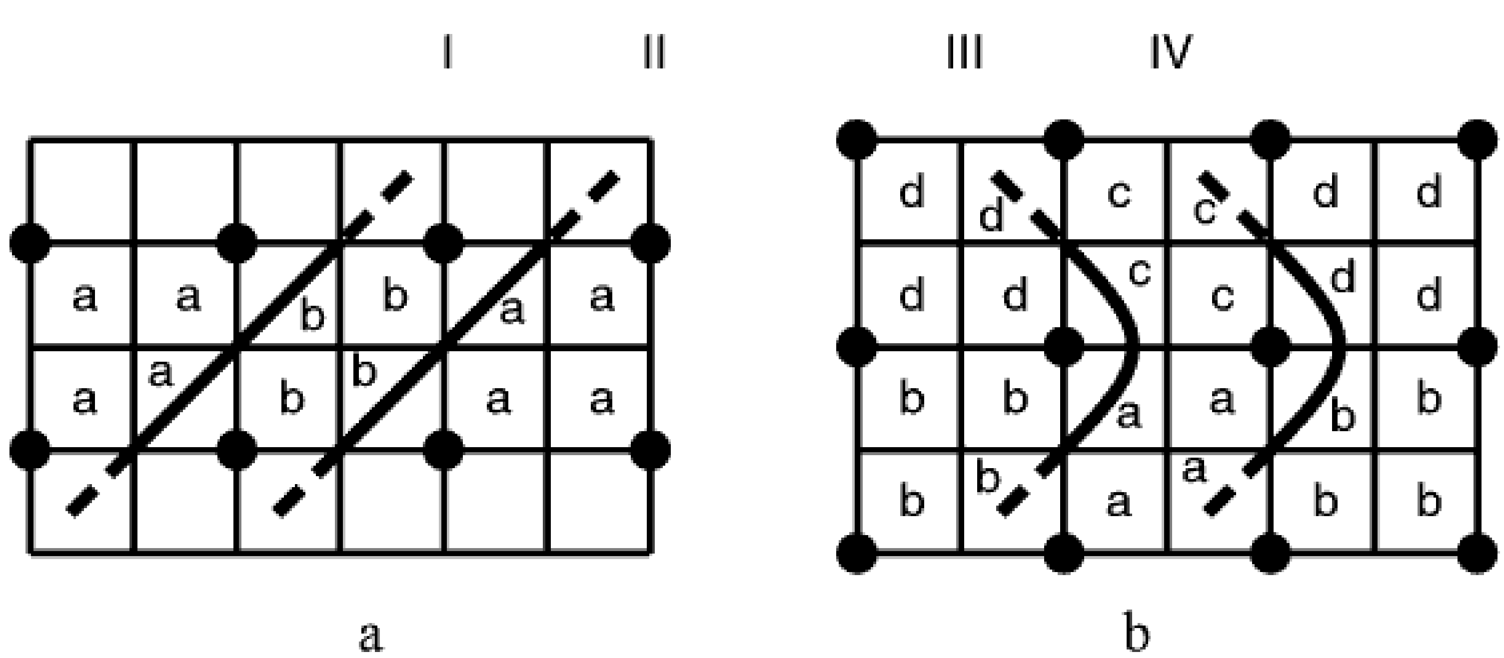


Figure 1.27

Corollary 2: Two crossing segments of a monolinear, regular and smooth mirror lines design are always traversed in the same direction (that is, both upwards or both downwards).

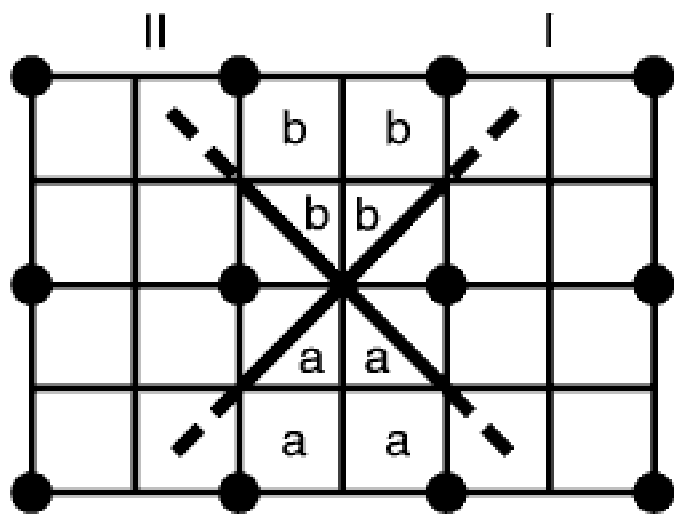
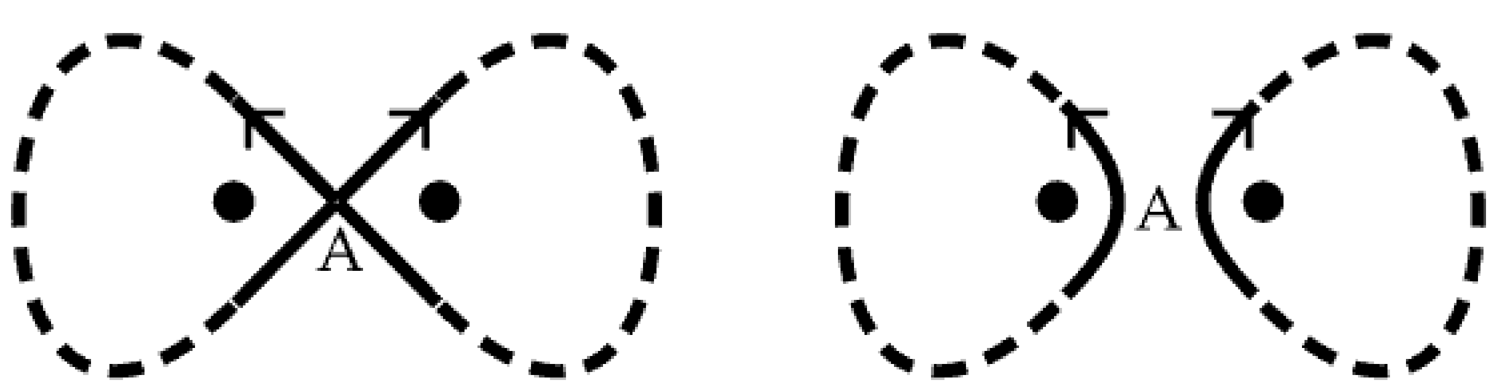


Figure 1.28

Proof:

If the segment I is traversed in the direction a*b, the segment II can only be traversed in the same direction (see Figure 1.28). In the case of b*a, the reasoning is the same.

Theorem 3: Take a monolinear, regular and smooth mirror lines design. If a crossing between two horizontal neighboring grid points is vertically eliminated (see Figure 1.29), a 2-linear mirror lines design is obtained.



Situation before the elimination

Situation after the elimination

Vertical elimination of a crossing between two horizontal neighboring grid points
Figure 1.29

Proof:

If one starts the course of the monolinear mirror curve from the crossing (A) to be eliminated onwards, climbing to the right (see Figure 1.29), then, continuing the course, one returns to A, in agreement with Corollary 2, from below on the right hand side; passes then through A and returns finally from below on the left hand side to A. This implies, if the curve is 'cut' in A, that two closed lines are obtained.

Figure 1.30 presents examples.

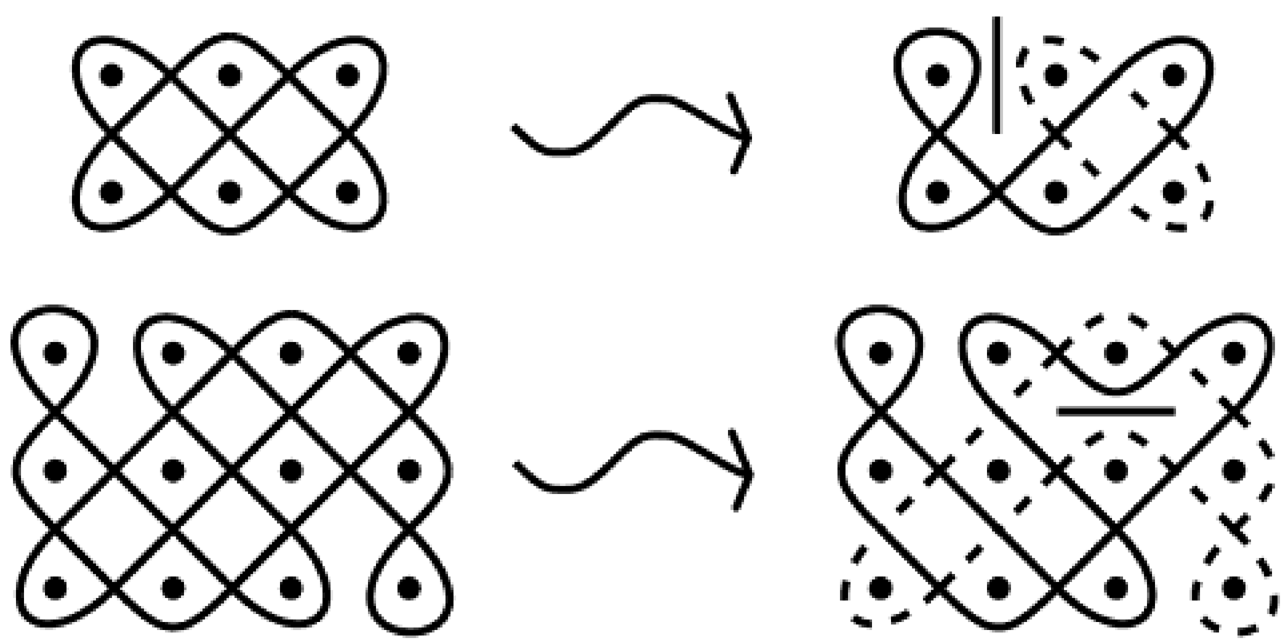


Figure 1.30

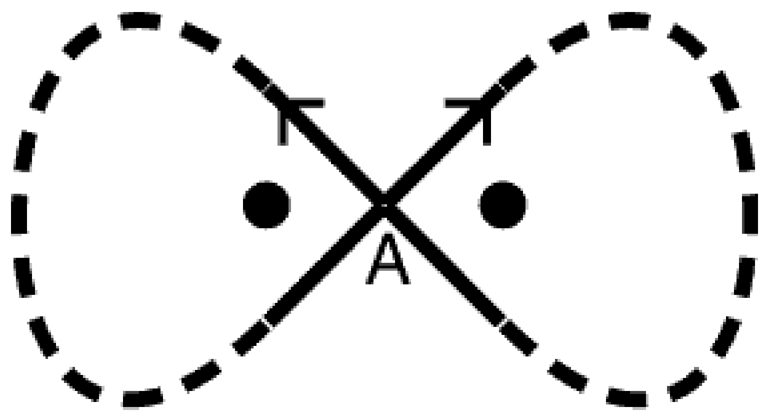
On symmetry grounds we have:

Corollary 3: Take a monolinear, regular and smooth mirror lines design. If a crossing between two vertical neighboring grid points is horizontally eliminated (see Figure 1.31), a 2-linear mirror lines design is obtained.

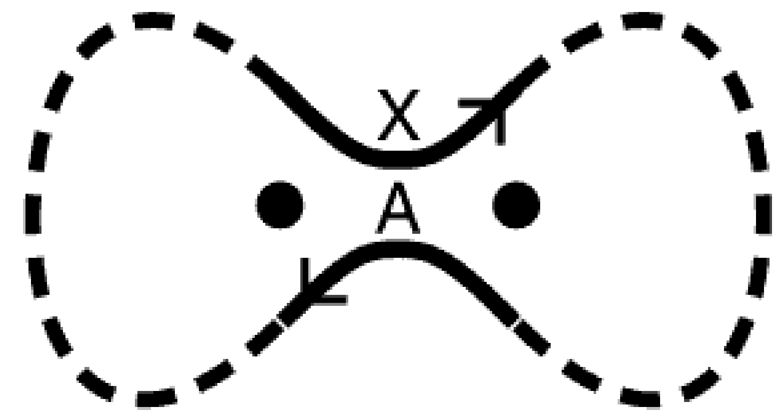
Situation before the elimination   Situation after the elimination

Horizontal elimination of a crossing
between two vertical neighboring grid points
Figure 1.31

Theorem 4: Consider a monolinear, regular and smooth mirror lines design. If a crossing between two horizontal neighboring grid points is horizontally eliminated, the mirror lines design thus obtained also is monolinear.



a: Situation before the elimination



b: Situation after the elimination

Horizontal elimination of a crossing
between two horizontal neighboring grid points
Figure 1.32

Proof:

Let us observe the given mirror lines design and traverse it from the crossing (A) to be eliminated on, upwards to the right (see Figure 1.32a).

Once more we have, in agreement with Corollary 2, that one returns to A from below on the right hand side.

Conversely, if we traverse the mirror lines design, starting in A, going downwards to the left, one returns to A once again from below on the left hand side, in agreement with Corollary 2.

Let us now eliminate horizontally the crossing A and traverse the mirror lines design from X (see Figure 1.32b) onwards in the indicated direction (-->). As the initial mirror lines design was monolinear, at a given moment one traverses the arc below A, from the right to the left. The monolinearity of the initial mirror lines design implies, taking into account Corollary 2, that one finally returns from the left to the start point X, having gone through the whole mirror lines design. Thus the proof of the theorem has been concluded.

For reasons of symmetry we have:

Corollary 4: Consider a monolinear, regular and smooth mirror lines design. If a crossing between two vertical neighboring grid points is vertically eliminated, the mirror lines design thus obtained is also monolinear.

Figure 1.33 gives examples.

It should be noted that the resulting new mirror lines designs are not regular and, therefore, the same theorem cannot be applied to them.

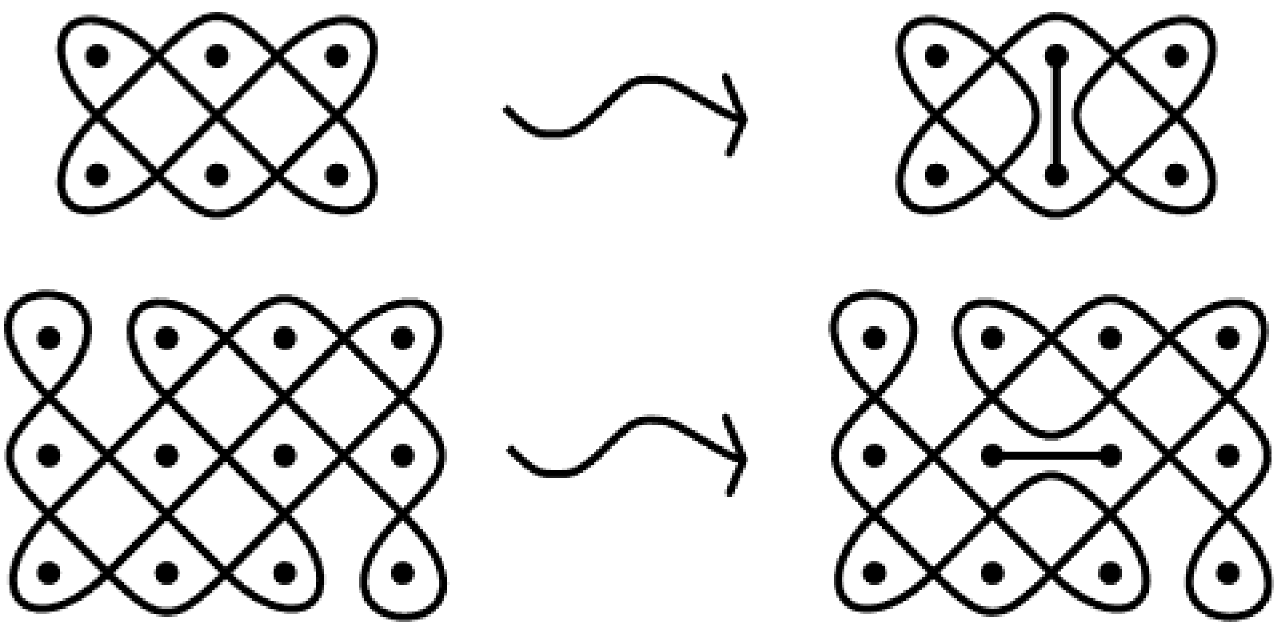


Figure 1.33

1.4 Enumeration modulo 2 and Lunda-designs

Theorem 2 gives information about the distribution of the 0's, 1's, 2's and 3's when one enumerates modulo 4 the unit squares through which a regular (monolinear) mirror curve successively passes. On the basis of this distribution we may deduce the distribution of 0's and 1's when we count them modulo 2 instead of modulo 4. Figure 1.34 gives an example.

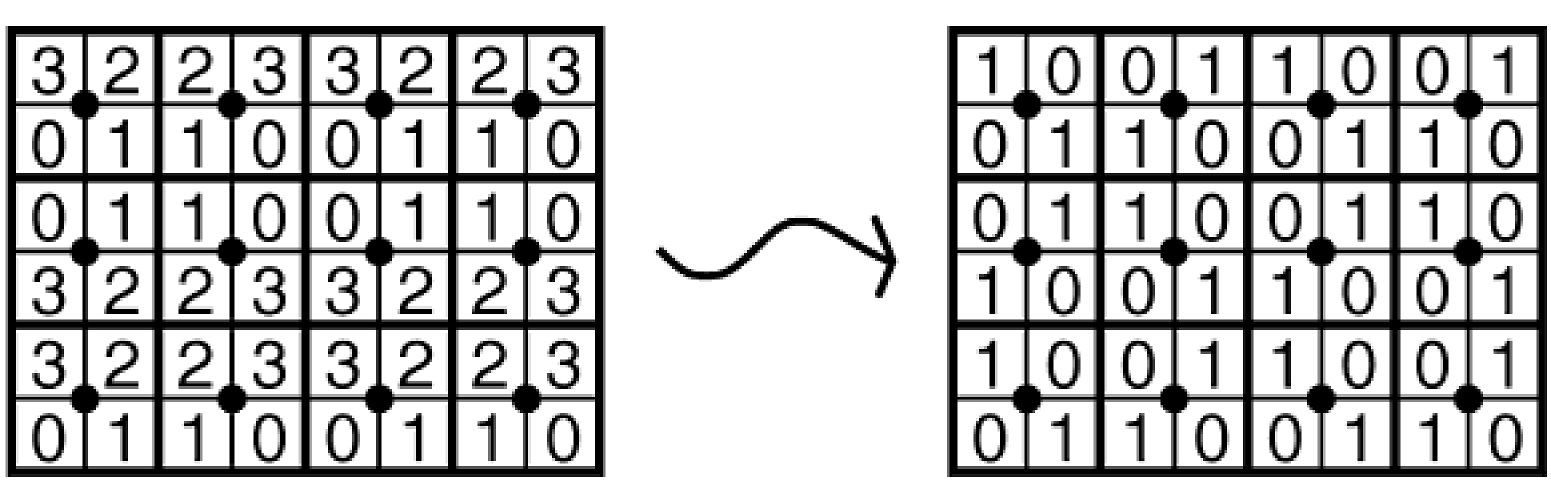


Figure 1.34

Coloring the unit squares with number 1 black, and the ones with number 0 white, black-and-white designs are obtained of the type illustrated in Figure 1.35, which corresponds to the example of the previous figure.

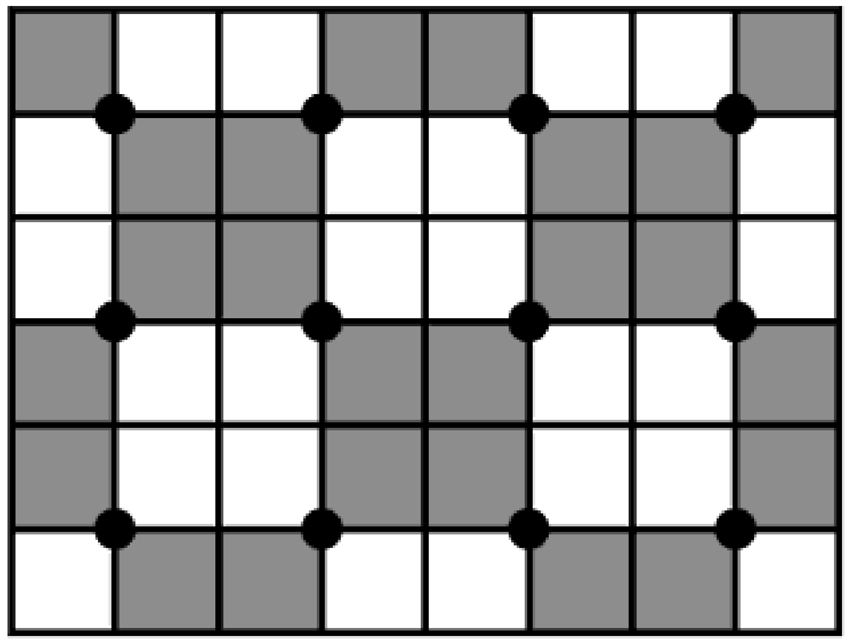


Figure 1.35

Non-regular mirror curves generate other distributions of 0’s, 1’s, 2’s and 3’s and other black-and-white designs. Figure 1.36 presents examples of dimensions 4 by 3:

1. Position of the mirrors;
2. Corresponding mirror curves;
3. Corresponding 0, 1, 2 and 3 designs;
4. Corresponding black-and-white designs.

As this type of black-and-white design was discovered in the context of analyzing sand drawings from the Cokwe, who predominantly inhabit the northeastern part of Angola, a region called **Lunda**, I have given them the name of **Lunda-designs**. For the first time I presented Lunda-designs in a paper published in 1990.

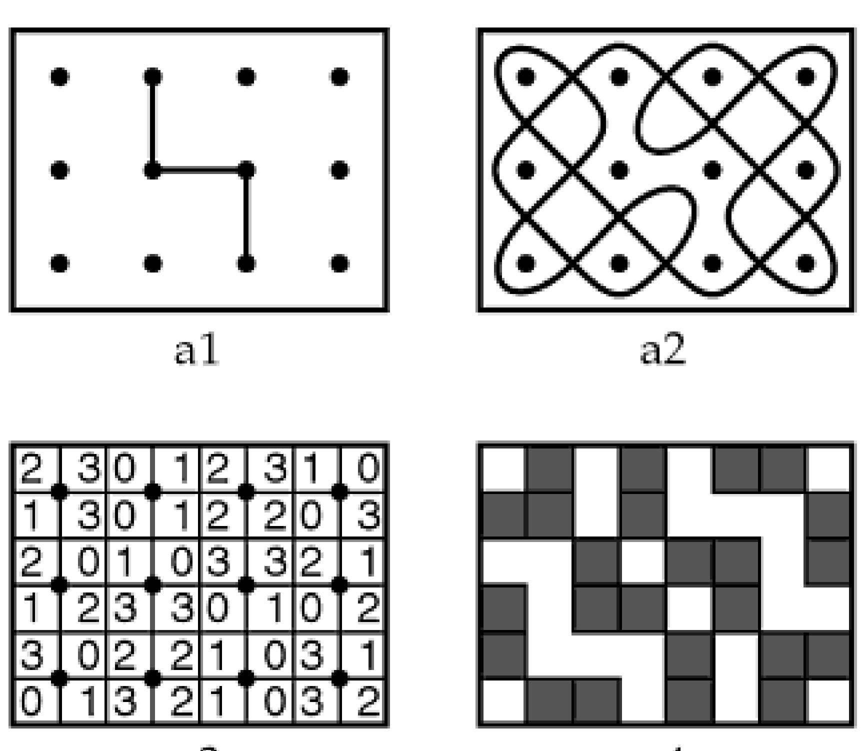


Figure 1.36 (first part)

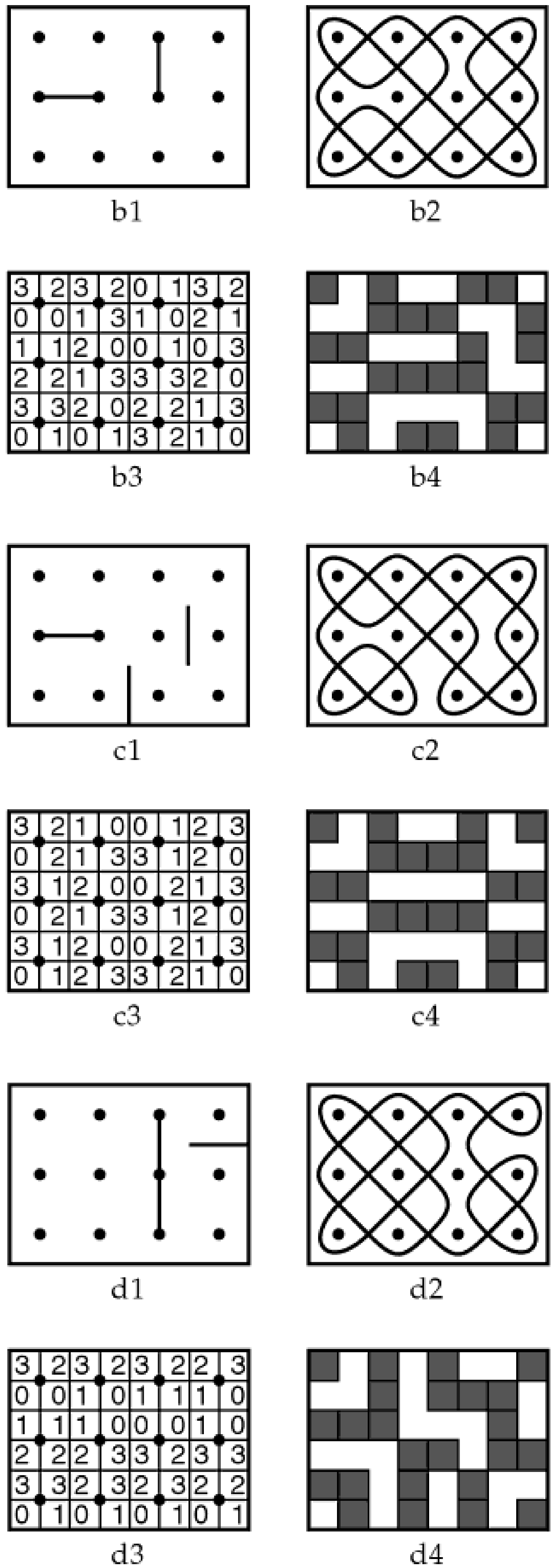


Figure 1.36 (continued)

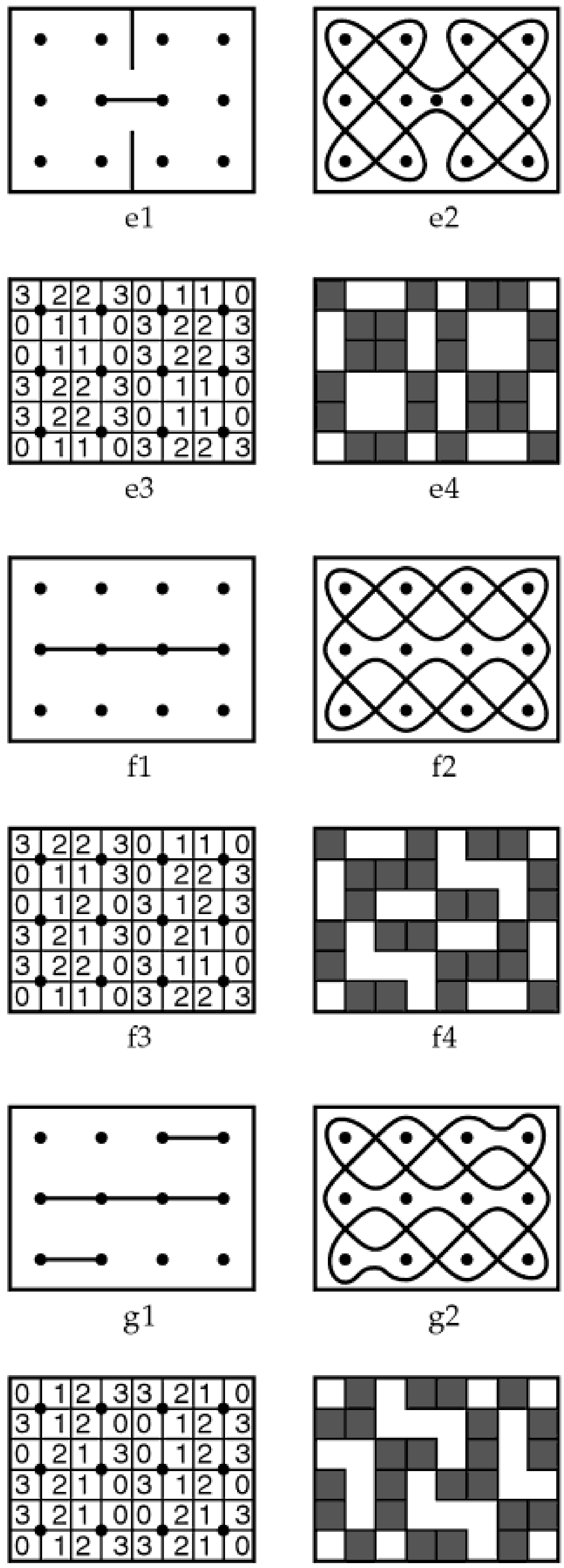


Figure 1.36 (continued)

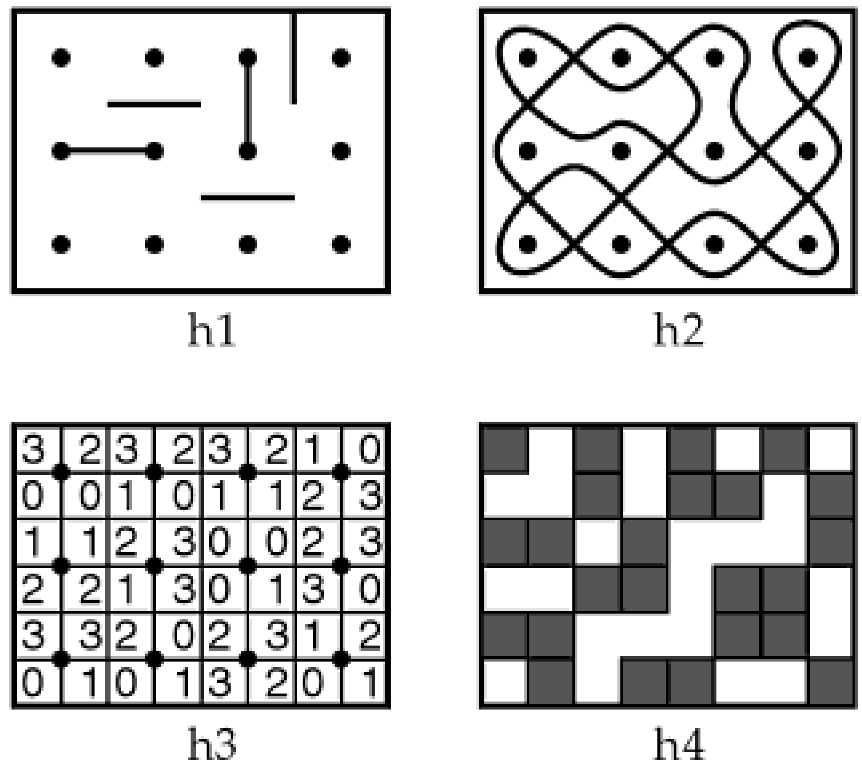


Figure 1.36 (conclusion)

References

Gerdes, Paulus (1990), On ethnomathematical research and symmetry, *Symmetry: Culture and Science*, Budapest, Vol.1, Nº 2, 154-170

___ (1993-94), *Geometria Sona: Reflexões sobre uma tradição de desenho em povos da África ao Sul do Equador*, ISP, Maputo (3 volumes); French translation, *Une tradition géométrique en Afrique — Les dessins sur le sable*, 3 volumes, L'Harmattan, Paris, 1995, 594 pp.; German translation, *Ethnomathematik am Beispiel der Sona Geometrie*, Spektrum Verlag, Heidelberg, 1997, 436 pp.; English translation of Vol.1, *Sona geometry: Reflections on the tradition of sand drawings in Africa South of the Equator*, ISP, Maputo, 1994, 200 pp. (new edition: 2006).

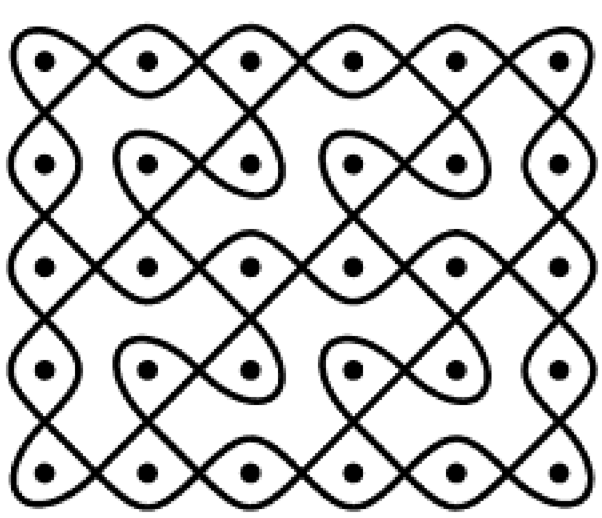
___ (2006), *Sona Geometry from Angola: Mathematics of an African Tradition*, Polimetrica International Science Publishers, Monza, 232 pp. [Preface by Arthur B. Powell].

Chapter 2

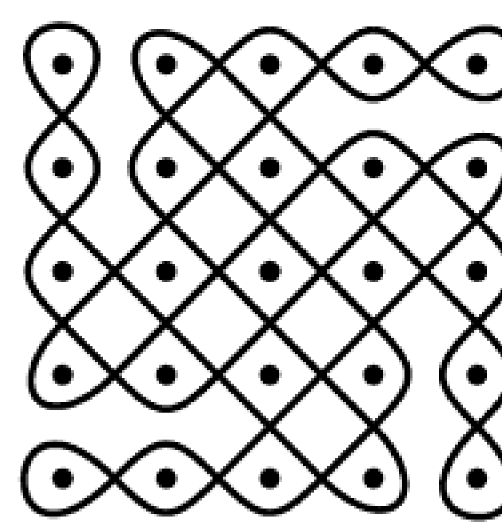
ON LUNDA-DESIGNS AND SOME OF THEIR SYMMETRIES [1]

2.1 Mirror designs and mirror curves

When analyzing sand drawings from the Cokwe (Angola) [cf. Gerdes, 1993-94] and threshold designs from the Tamil (South India) [cf. Gerdes, 1989; 1993-94, chap. 11; 1995], I found that several of them (see the two examples in Figure 2.1) might be generated in the following way.



Cokwe sand drawing
a

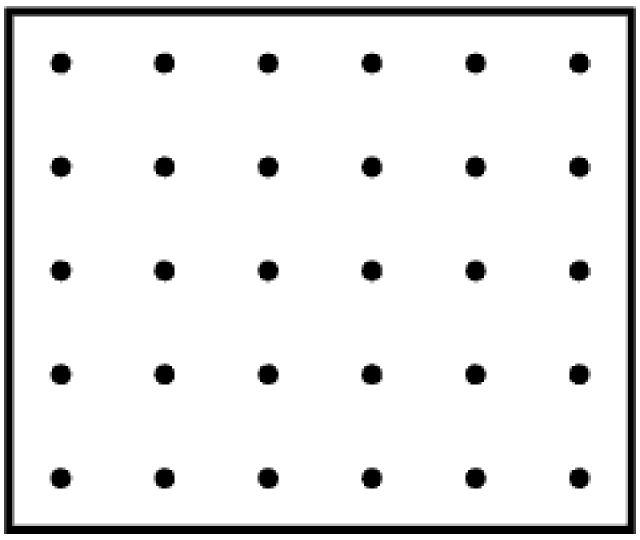


Tamil threshold design
b

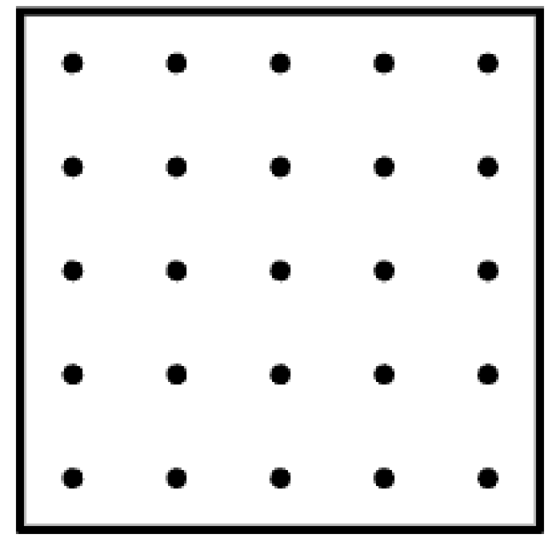
Figure 2.1

Consider a rectangular grid RG[m,n] with vertices (0,0), (2m,0), (2m,2n), and (0,2n) and having as points (2s-1, 2t-1), where s = 1,...,m, and t = 1,...,n, and m and n two arbitrary natural numbers. Figure 2.2 displays RG[6,5] and RG[5,5].

[1] Published in the electronic journal: *Visual Mathematics*, Belgrade, Vol. 1, No. 1, 1999.



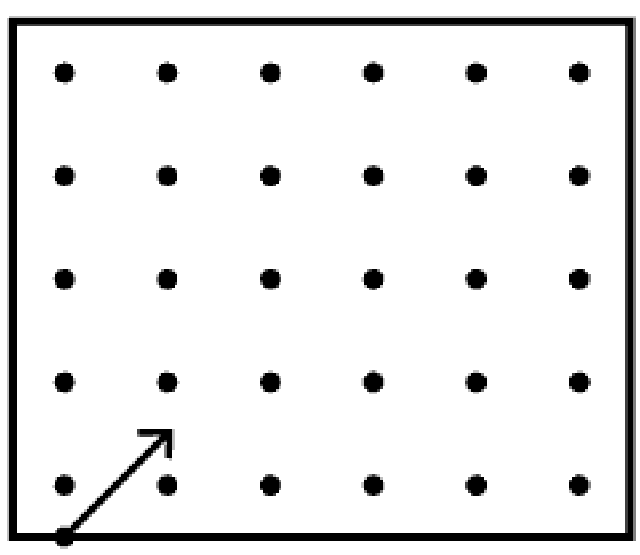
RG[6,5]



RG[5,5]

Figure 2.2

A curve like that shown in Figure 2.1 is the smooth version of a closed polygonal path described by a light ray emitted from the point (1,0) at an angle of 45^{o} to the sides of the rectangular grid RG[m,n] (see the example in Figure 2.3).

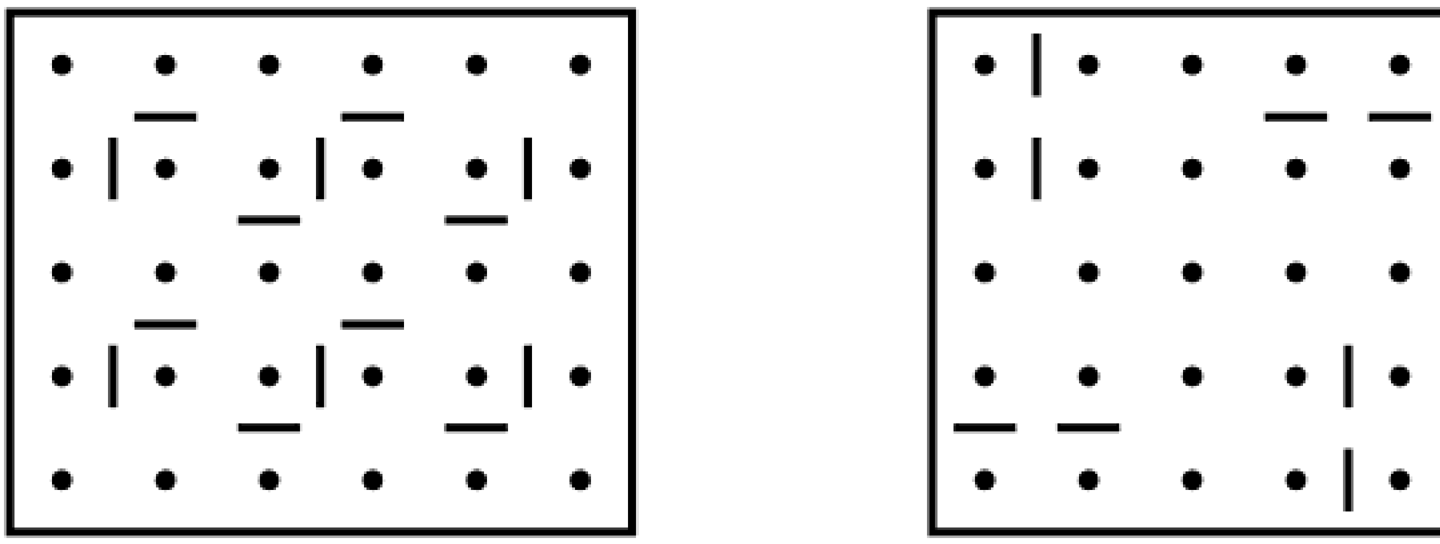


Emission of a light ray from the point (1,0)
Figure 2.3

The ray is reflected on the sides of the rectangle and on its way through the grid it encounters double-sided mirrors, which are placed horizontally or vertically, midway, between two neighboring grid points (see Figure 2.4).

Possible positions of mirrors
Figure 2.4

Figure 2.5 shows the position of the mirrors in order to generate the two curves of Figure 2.1.

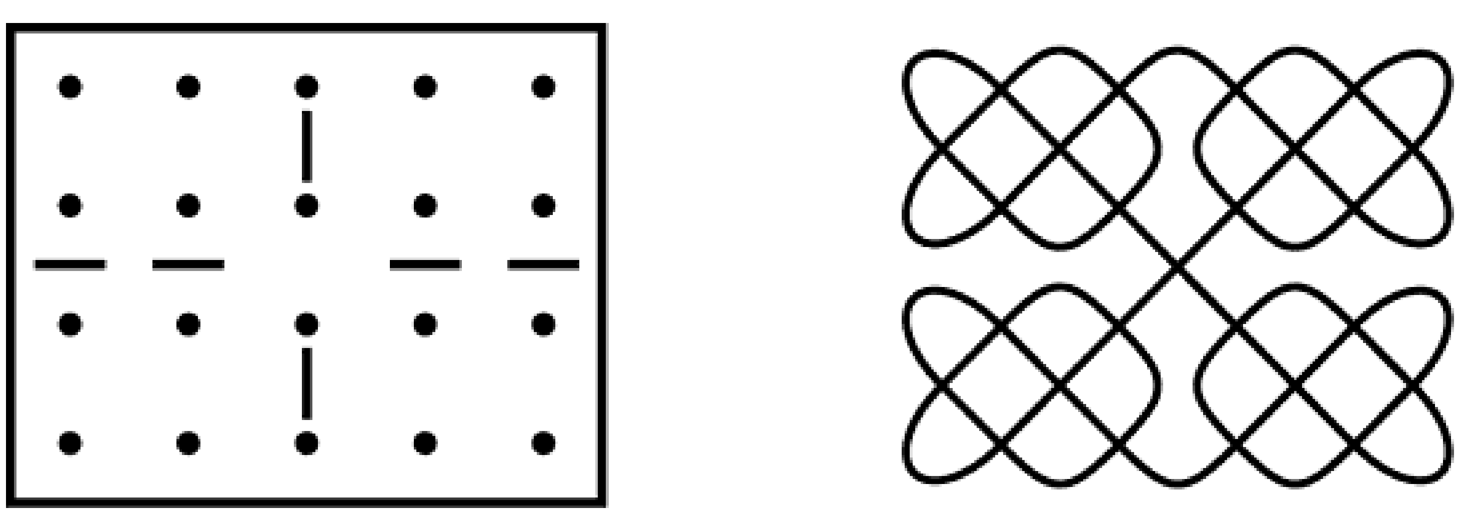


a b

Mirror designs generating the curves of Figure 2.1

Figure 2.5

Both curves are rectangle-filling in the sense that they 'embrace' all grid points. Such curves we will call (rectangle-filling) **mirror curves**. The rectangular grids together with the mirrors, which generate the curves will be called **mirror designs**. Figure 2.6a displays the mirror design that leads to the Celtic knot design in Figure 2.6b (cf. Gerdes, 1993-94, chap. 12).

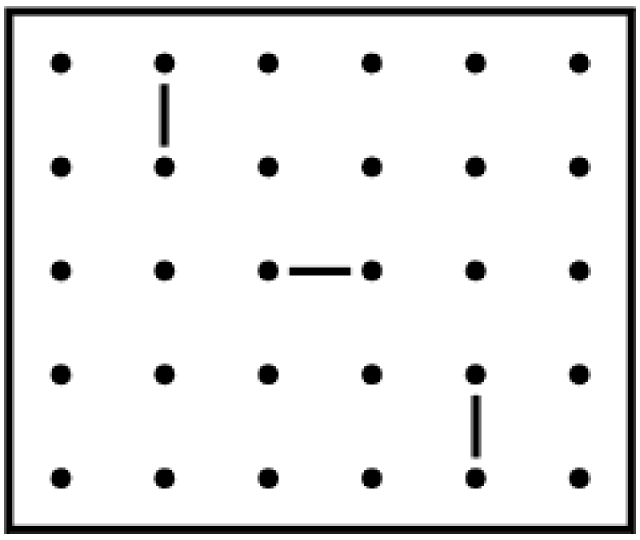


a b

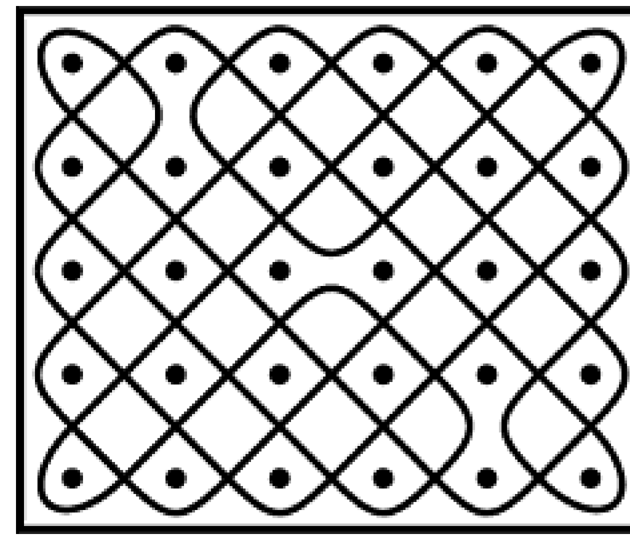
Example of a Celtic knot design as a mirror curve

Figure 2.6

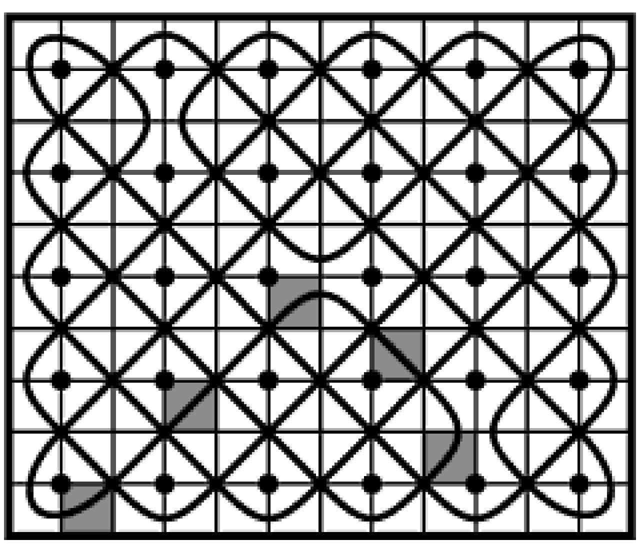
Gerdes (cf. 1990, 1993-94, chap. 4-8) analyzes some properties and classes of mirror curves and Jablan (1995) establishes links between mirror curves and the theory of cellular automata, Polya's enumeration theory, combinatorial geometry, topology, and knot theory.



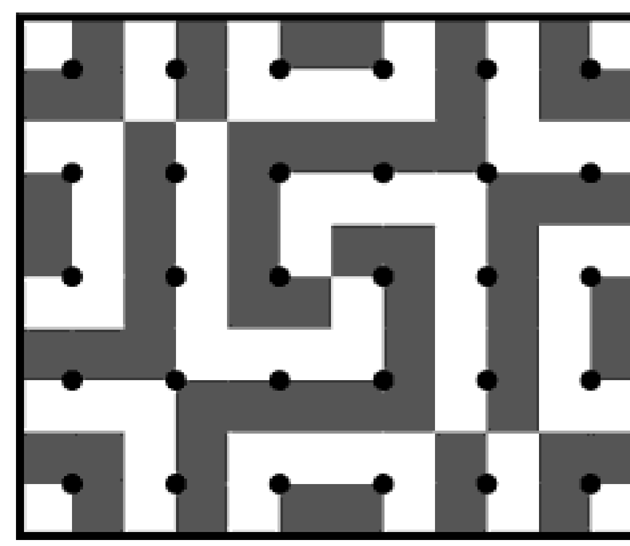
Mirror design
a



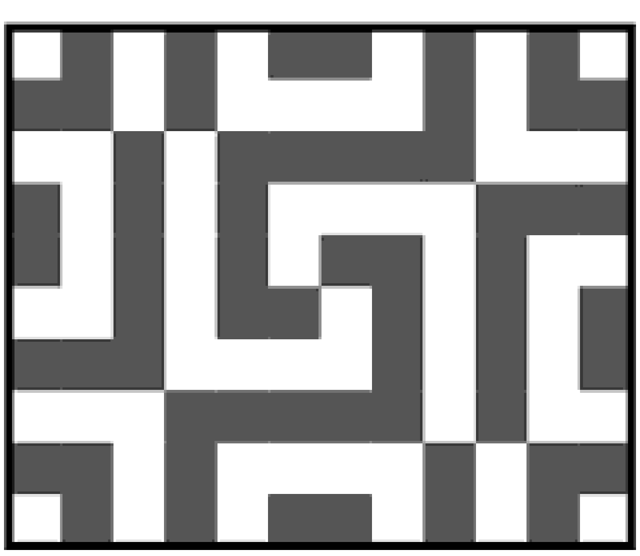
Corresponding mirror curve
b



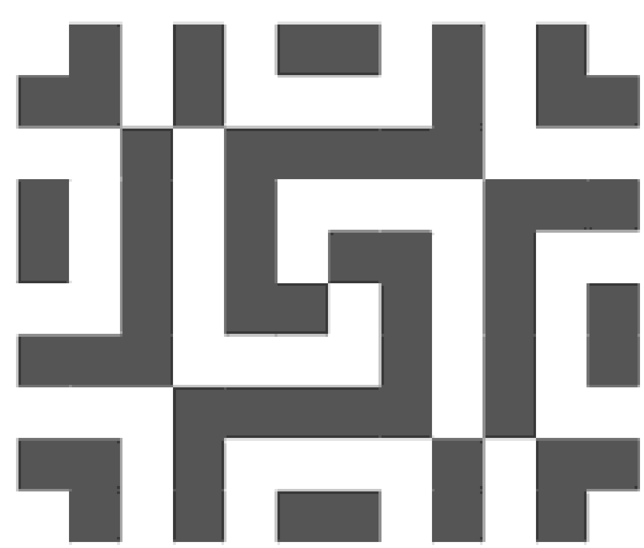
Starting the coloring
c



Final black-and-white pattern
(with the grid points marked)
d



Final black-and-white pattern
(grid points unmarked)
e



Final black-and-white pattern
(border rectangle unmarked)
f

Example of a black-and-white coloring
Figure 2.7

2.2 The discovery of Lunda-designs

Let us now consider a rectangle-filling mirror curve. It passes precisely once through each of the unit squares of the rectangular grid.

This enables us to enumerate the unit squares through which the curve successively passes, 1, 2, 3, 4,...,4mn. Enumerating them modulo 2, i.e. 1, 0, 1, 0, ..., 0, a (1,0)-matrix is obtained, or, equivalently, by coloring the successive unit squares alternately black (= 1) and white (= 0), a black-and-white design is produced. Figure 2.7 presents an example of the generation of such a black-and-white design.

As this type of black-and-white design was discovered in the context of analyzing sand drawings from the Cokwe, who predominantly inhabit the northeastern part of Angola, a region called **Lunda**, I have given them the name of **Lunda-designs**.

2.3 Examples

Figure 2.8b displays a sequence of nine 10x11 Lunda-designs generated by introducing, step-by-step, more horizontal mirrors along the principal diagonal (Figure 2.8a). Figure 2.9b shows what happens if we introduce the mirrors in pairs (Figure 2.9a). This time, the resulting Lunda-designs have a two-color symmetry: a half-turn about the centre interchanges white and black. This also happens with the Lunda-designs in Figure 2.10.

Figure 2.11a shows a sequence of three mirror designs, of which the second and third generate the same Lunda-design (Figure 2.11b). These Lunda-designs admit vertical and horizontal reflections. The first preserves the colors, whereas the second reverses black and white.

Figure 2.12a displays three mirrors designs with two-fold rotational symmetry. The Lunda-design generated by the first is also invariant under a half-turn about its centre. In the second and third cases, a half-turn around the respective centers reverses the colors.

The symmetrical mirror designs in Figure 2.13a generate Lunda-designs with horizontal and vertical reflections, which interchange black and white.

(Many) Lunda-designs seem to me — and to colleagues and students to whom I have shown them — aesthetically appealing. Where do possible reasons for this lie? What do all these Lunda-designs have in common? Which characteristics? Do they possess specific symmetries?

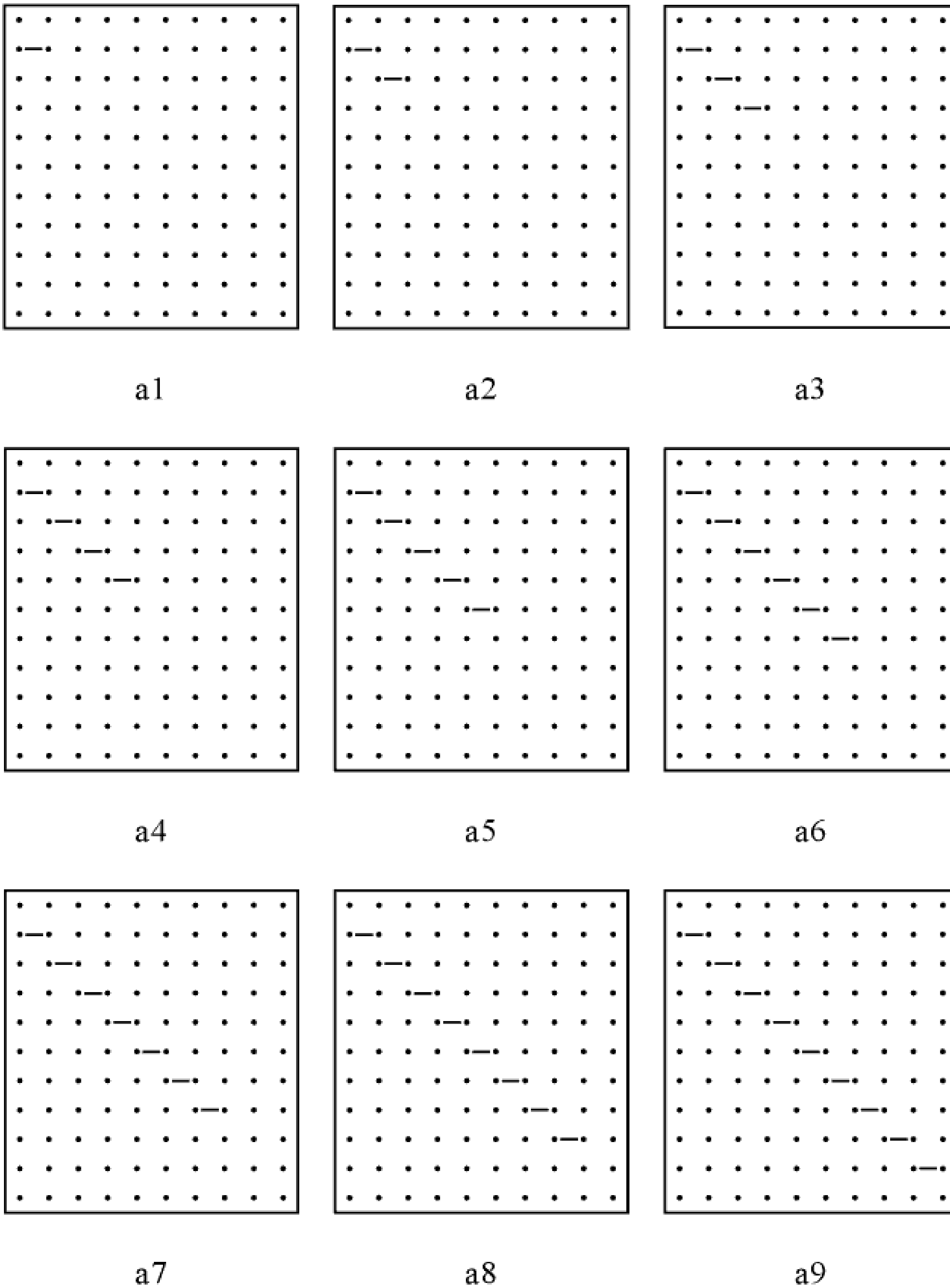
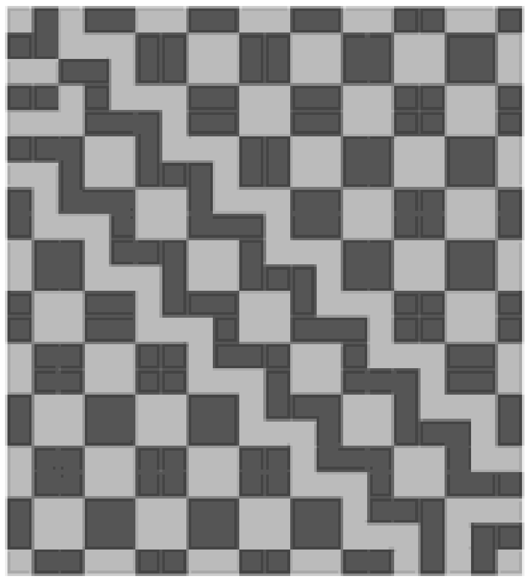
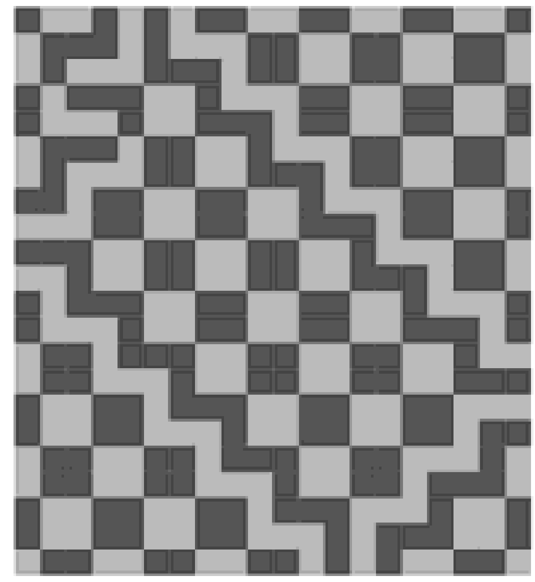


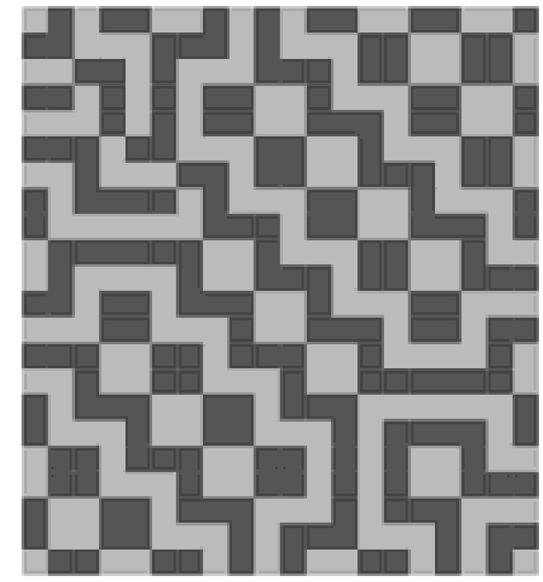
Figure 2.8 (first part)



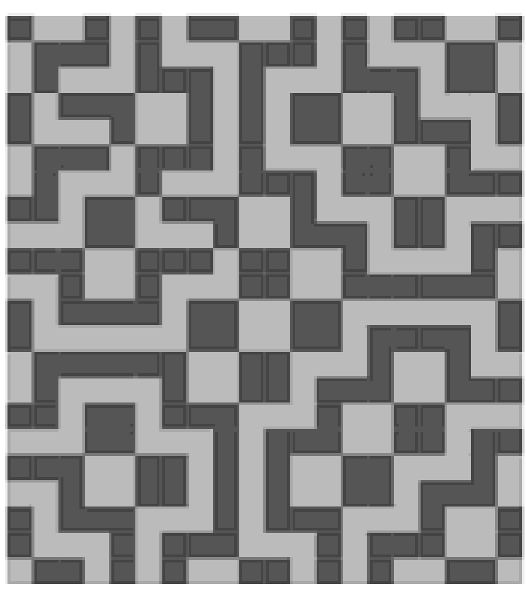
b1



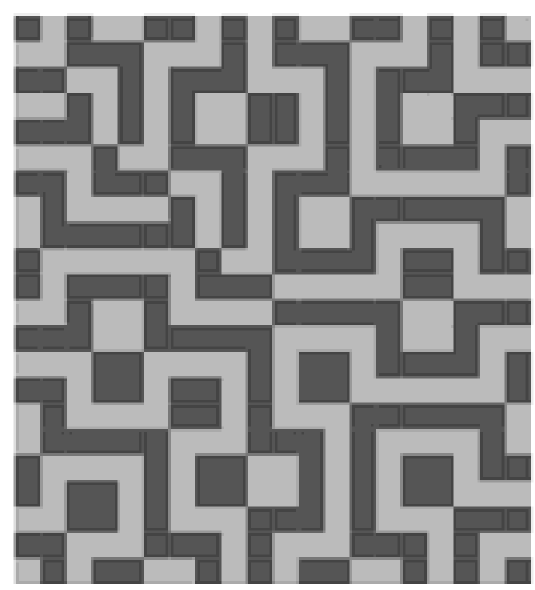
b2



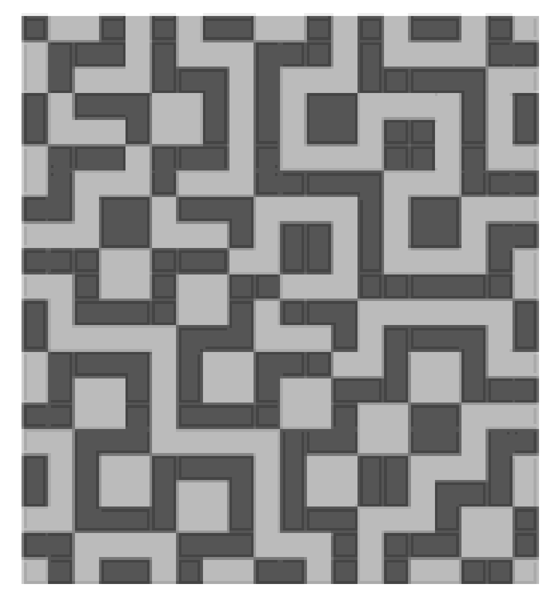
b3



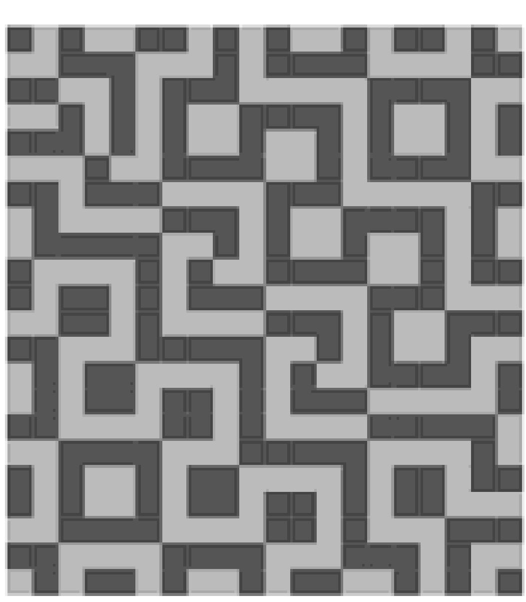
b4



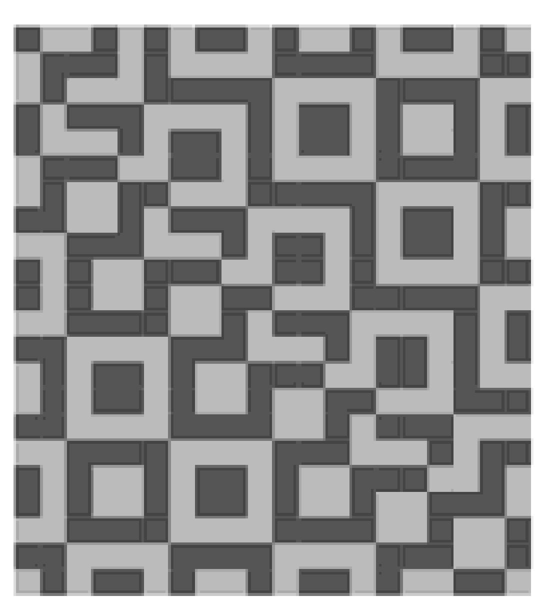
b5



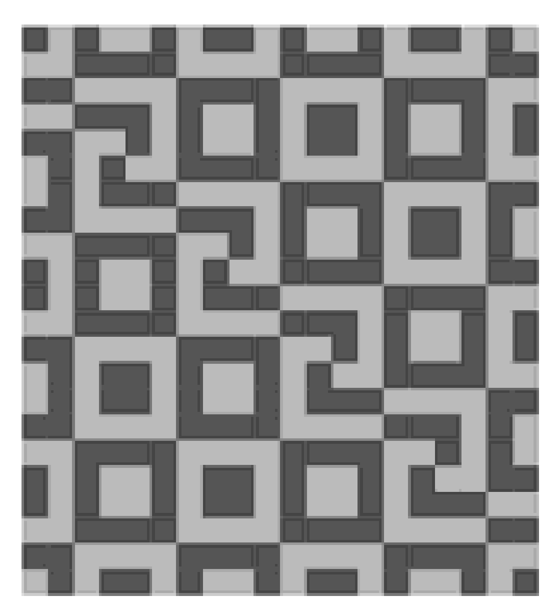
b6



b7



b8



b9

Figure 2.8 (second part)

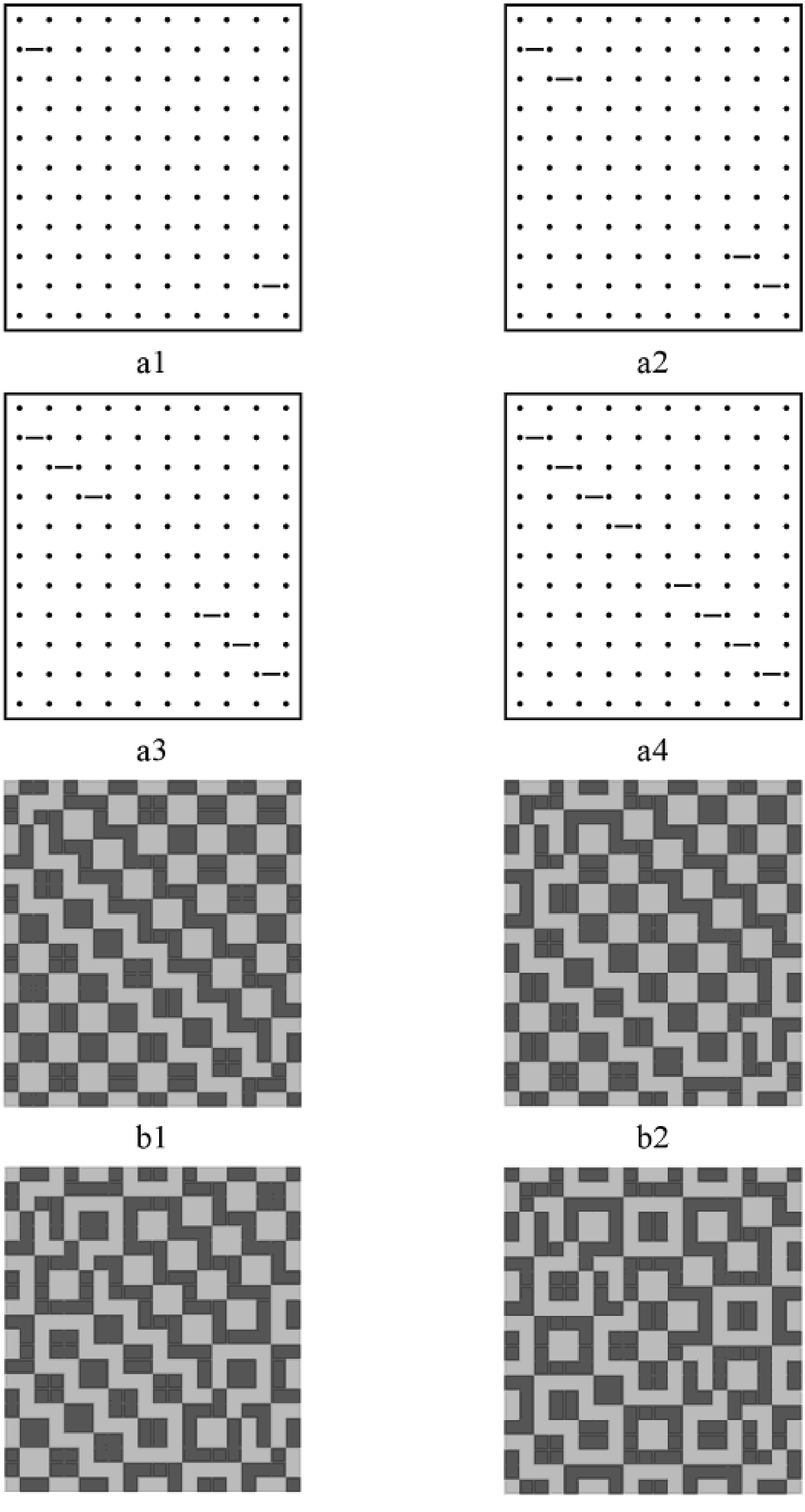
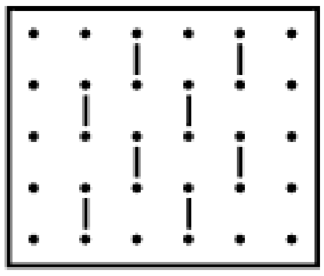
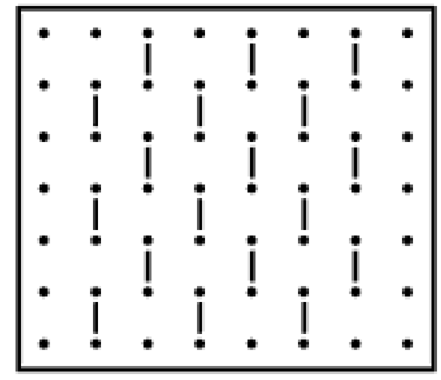


Figure 2.9



a1



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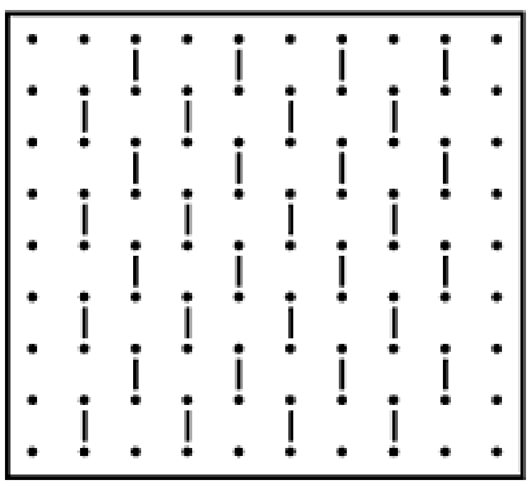

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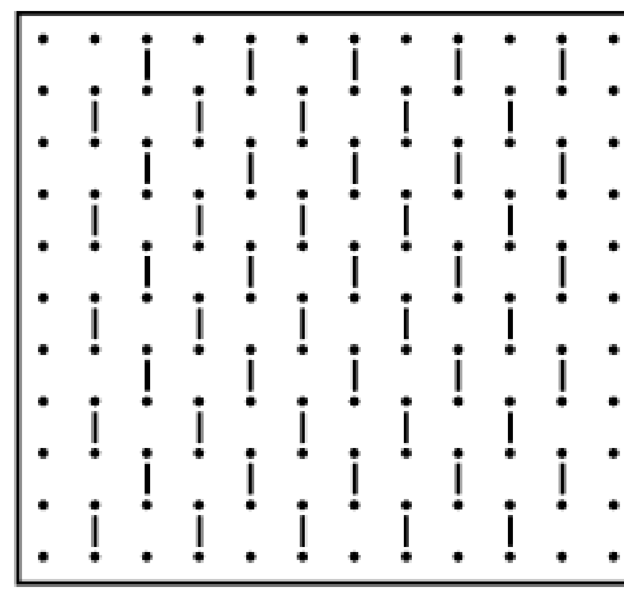

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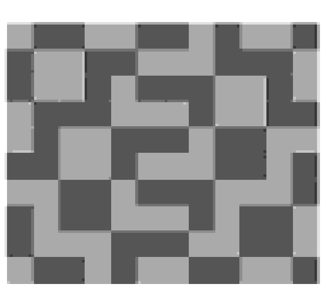

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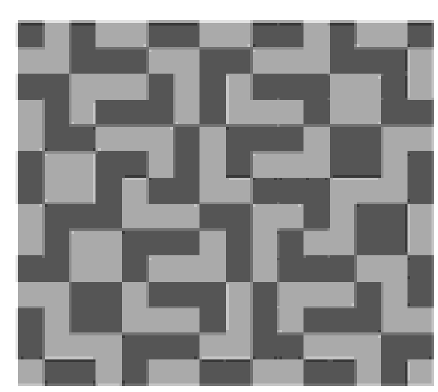

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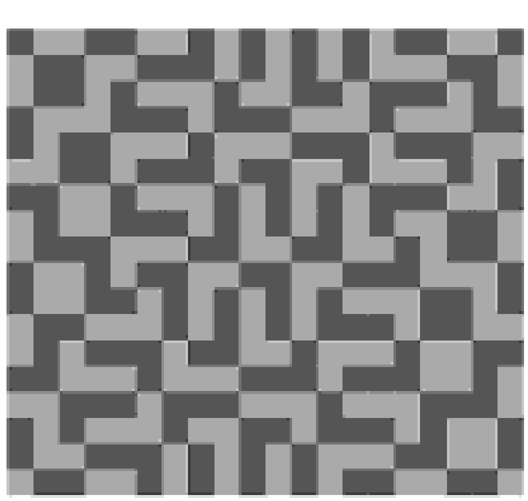

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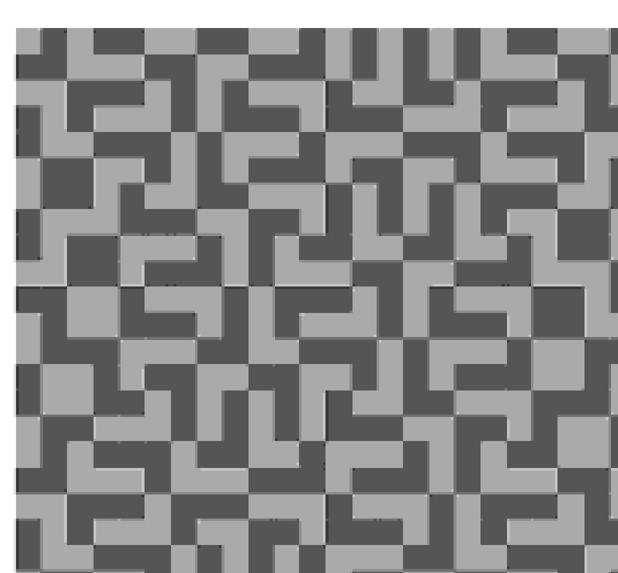

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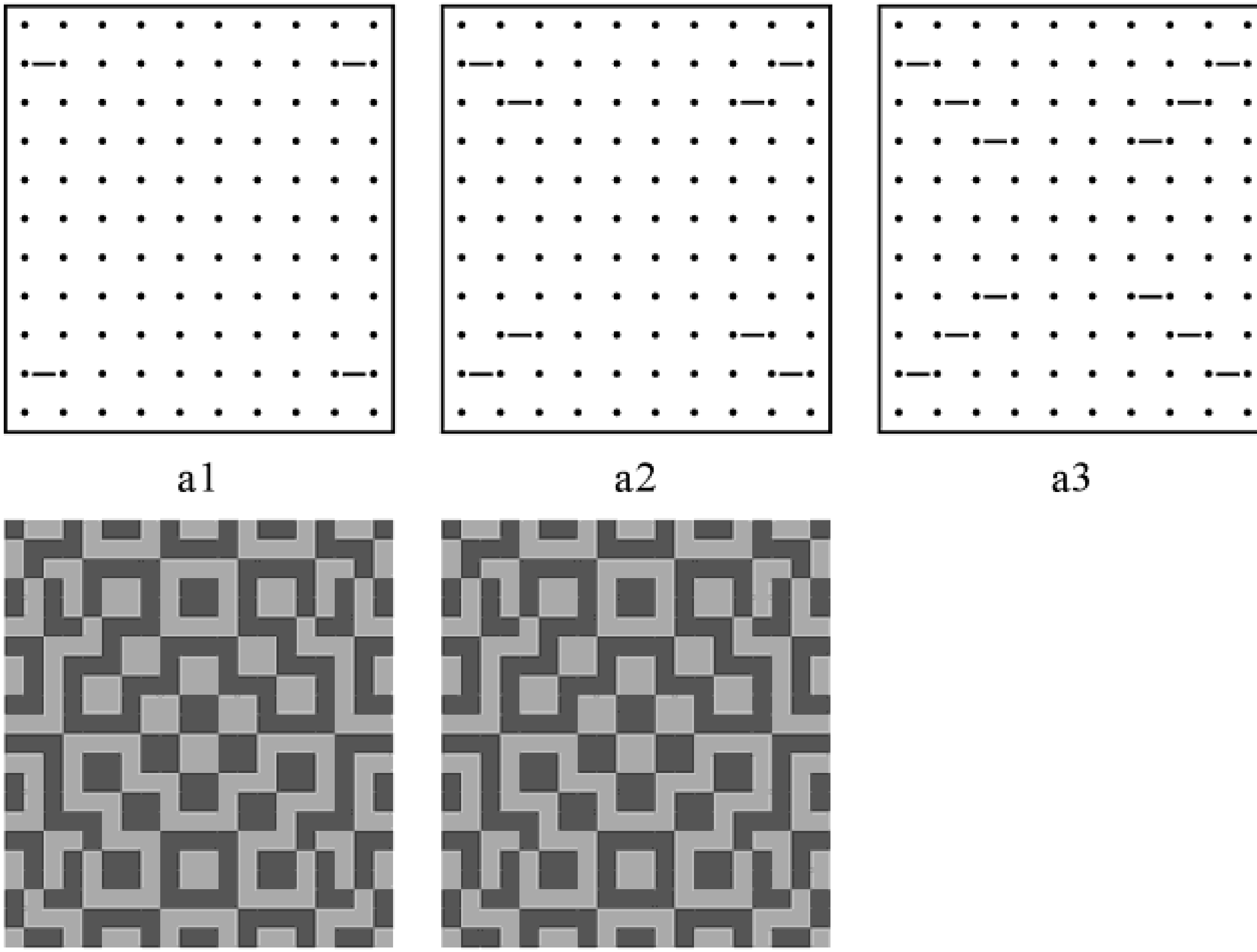
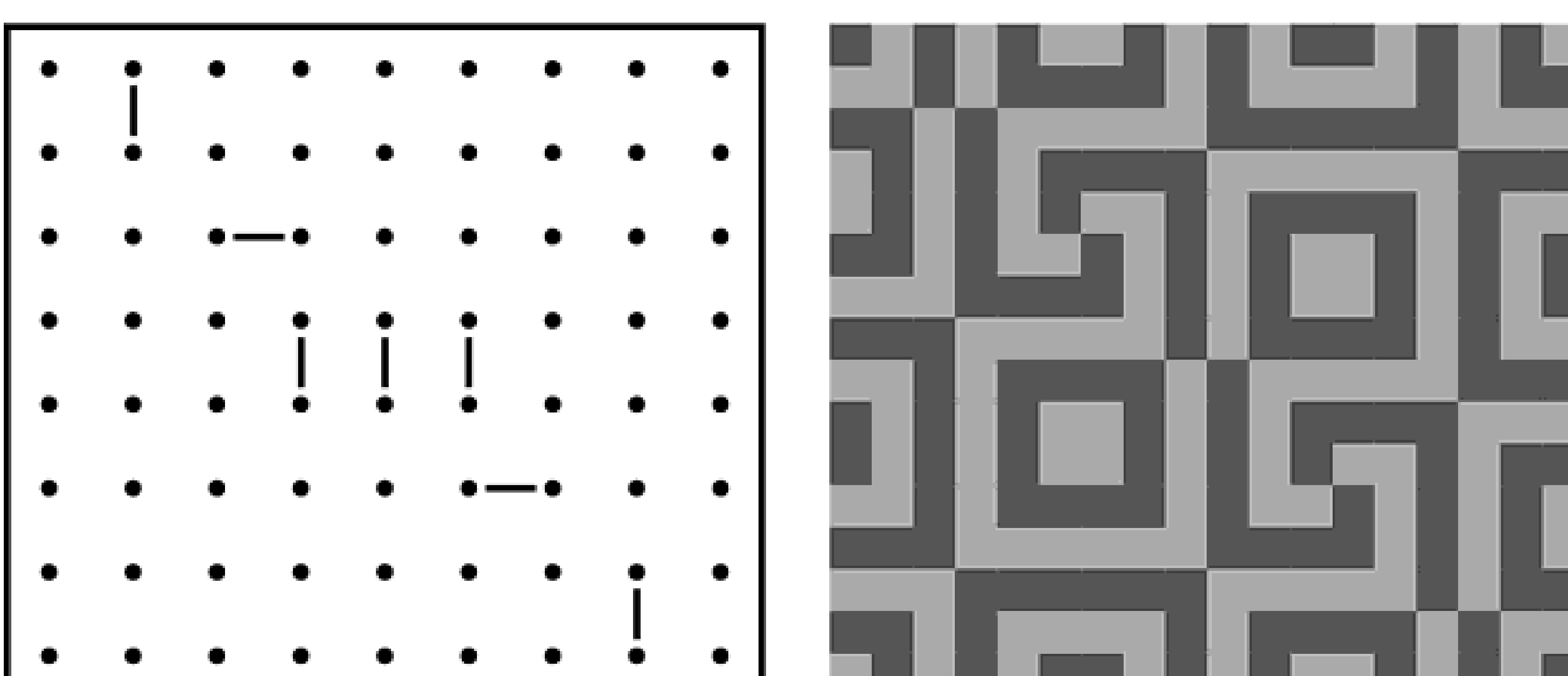
Figure 2.10



a1 a2 a3

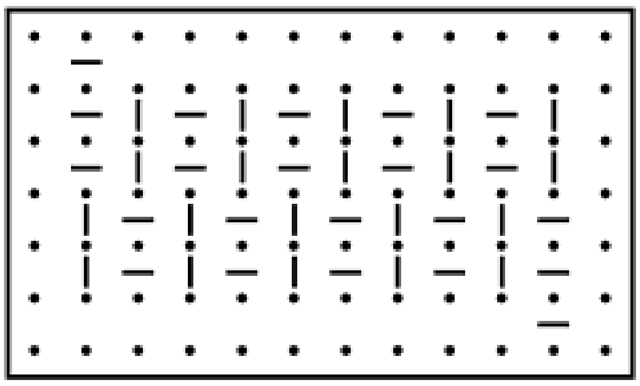
b1 b2 = b3

Figure 2.11



a1 b1

Figure 2.12 (First part)



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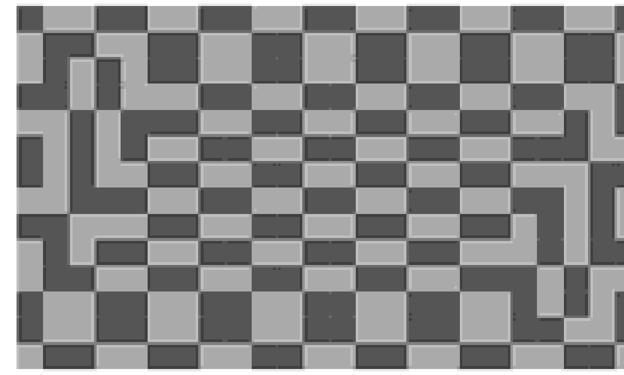

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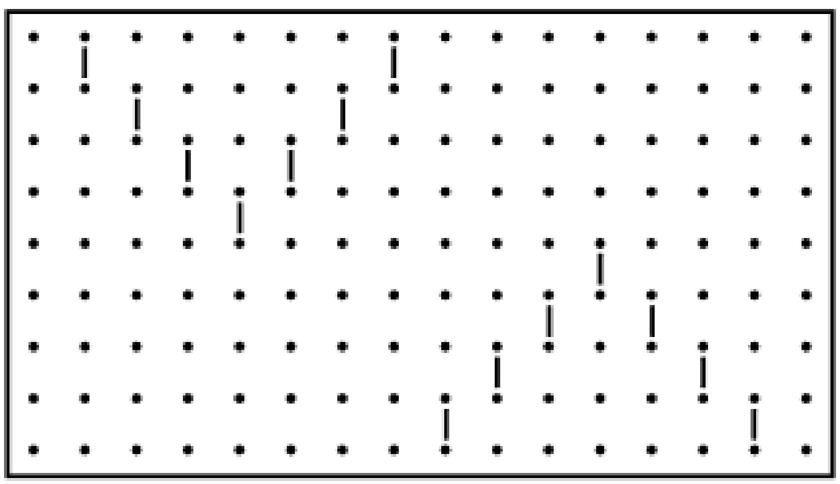

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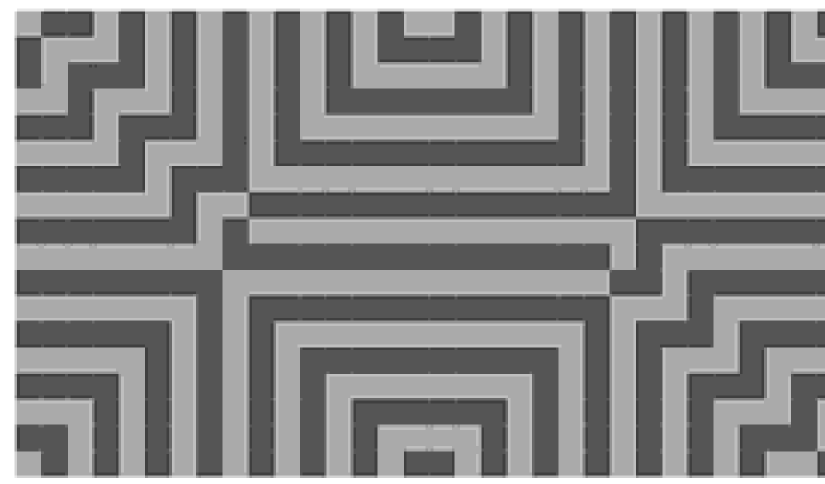
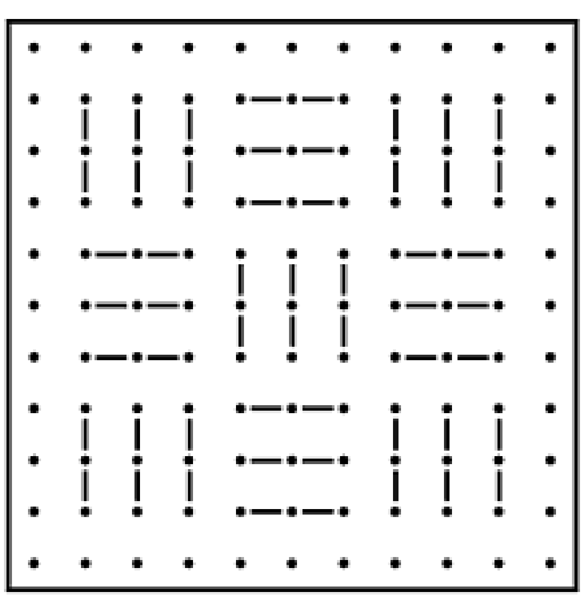


b3

Figure 2.12 (Second part)



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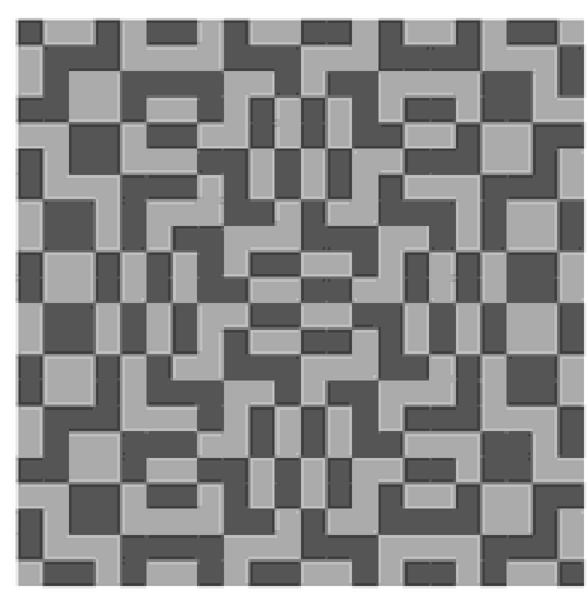

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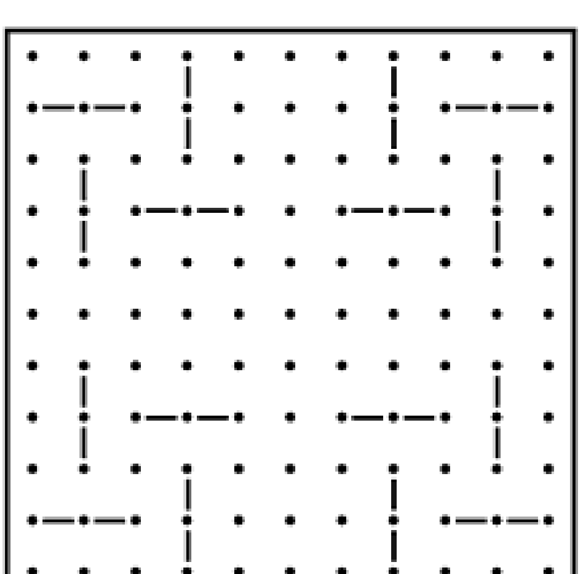

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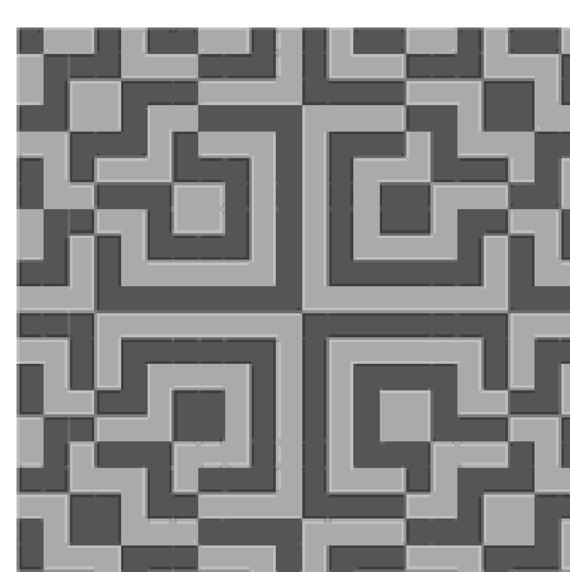


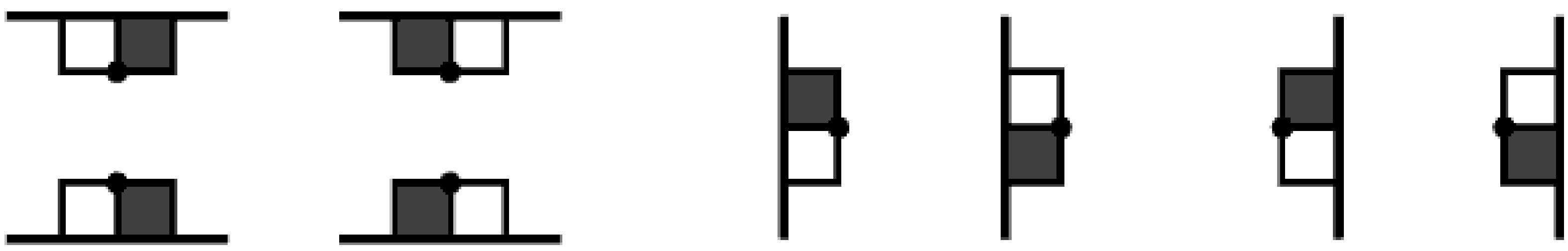
b2

Figure 2.13

2.4 General symmetry properties

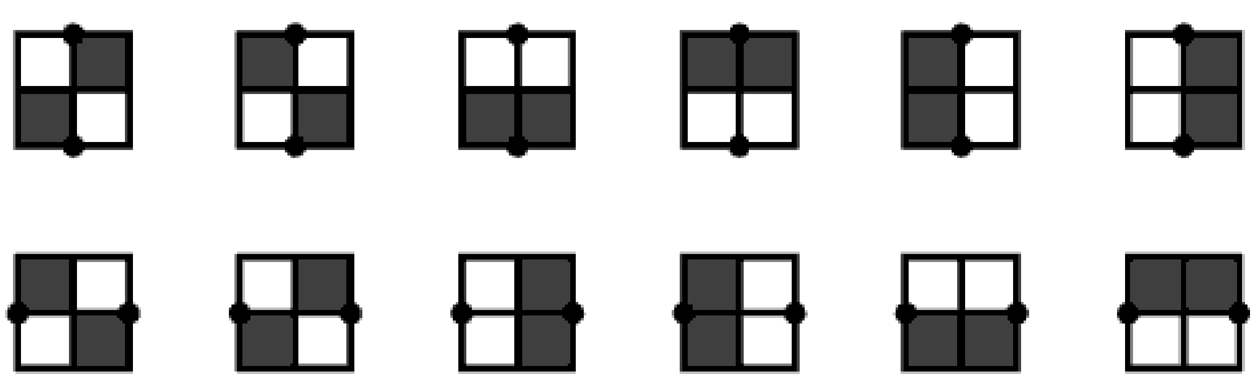
Searching for the common characteristics of Lunda-designs (of dimensions mxn), the following symmetry properties were observed and proven:

(i) In each row there are as many black as white unit squares;
(ii) In each column there are as many black as white unit squares;
(iii) Of the two border unit squares of any grid point in the first or last row, or in the first or last column, one is always white and the other black (see Figure 2.14);



Possible border situations
Figure 2.14

(iv) Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are always black and two are white (see Figure 2.15).



Possible situations between vertical and horizontal neighboring grid points
Figure 2.15

Properties (i) and (ii) guarantee a global equilibrium between black and white unit squares for each row and column. Properties (iii) and (iv) guarantee more local equilibriums.

From (i) it follows that the number of black unit squares of any row is equal to m, and from (ii) that the number of black unit squares of any column is equal to n.

Inversely, the following theorem can be proven:

* any rectangular black-and-white design that satisfies the properties (i), (ii), (iii), and (iv) is a Lunda-design.

In other words, for any rectangular black-and-white design that satisfies the properties (i), (ii), (iii), and (iv), there exists a (rectangle-filling) mirror curve that produces it in the discussed sense (cf. Figure 2.7). Moreover, in each case, such a mirror curve may be constructed.

The characteristics (i), (ii), (iii), and (iv) may be used to define Lunda-designs of dimensions mxn (we may abbreviate: **mxn Lunda-designs**). In fact, it may be proven that the characteristics (iii) and (iv) are sufficient for this definition, as they imply (i) and (ii) (see Appendix 1).

2.5 Special classes of Lunda-designs

Especially attractive are Lunda-designs, which display extra symmetries.

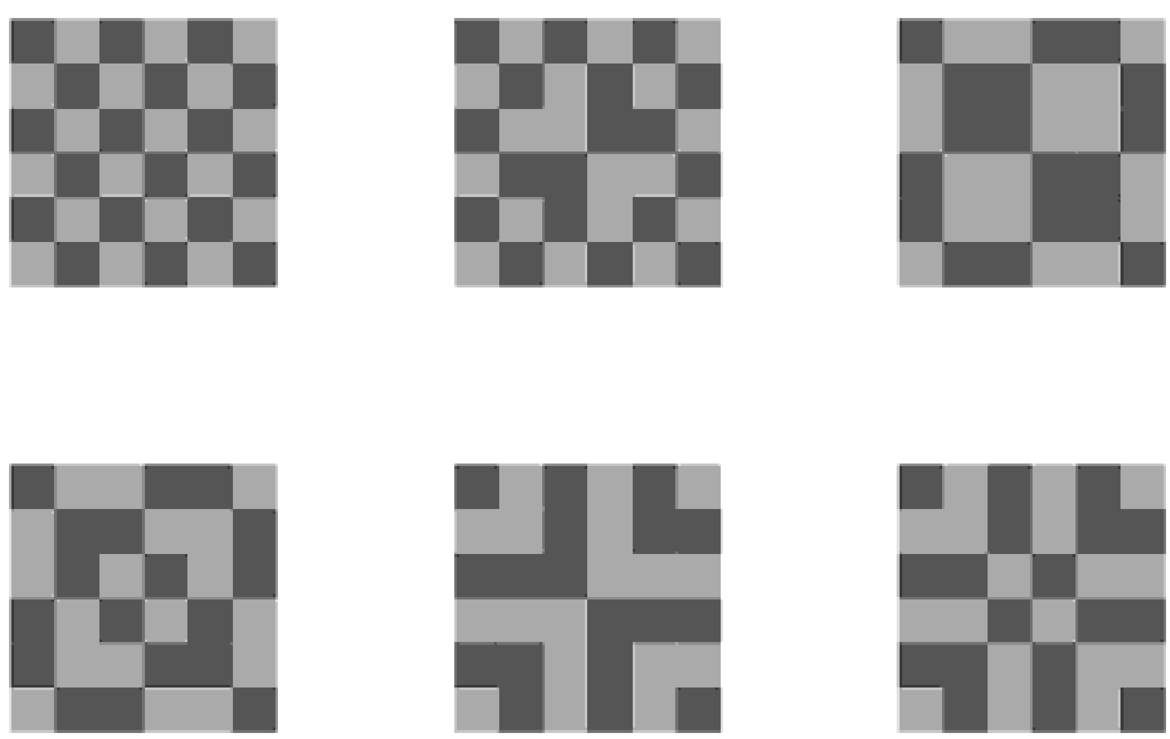


Figure 2.16

Figure 2.16 presents the six possible 3x3 Lunda-designs [being white (= 0) the color of the first unit square with vertices (0,0), (1,0), (1,1), and (0,1)] that admit reflections in the diagonals that preserve the colors and vertical and horizontal reflections interchanging black and white. By consequence, a half-turn about the centre preserves the colors and a quarter-turn reverses the colors. In other words, these finite designs are of the type *d4'* (for this notation, see e.g. Washburn & Crowe, p. 68).

Figure 2.17 displays the 4x4 Lunda-designs and Figure 2.18 the 5x5 Lunda-designs, which have the same symmetries. Figure 2.19 presents examples of mirror designs, which generate such 5x5 Lunda-designs (the numbers indicate the corresponding Lunda-designs in Figure 2.18).

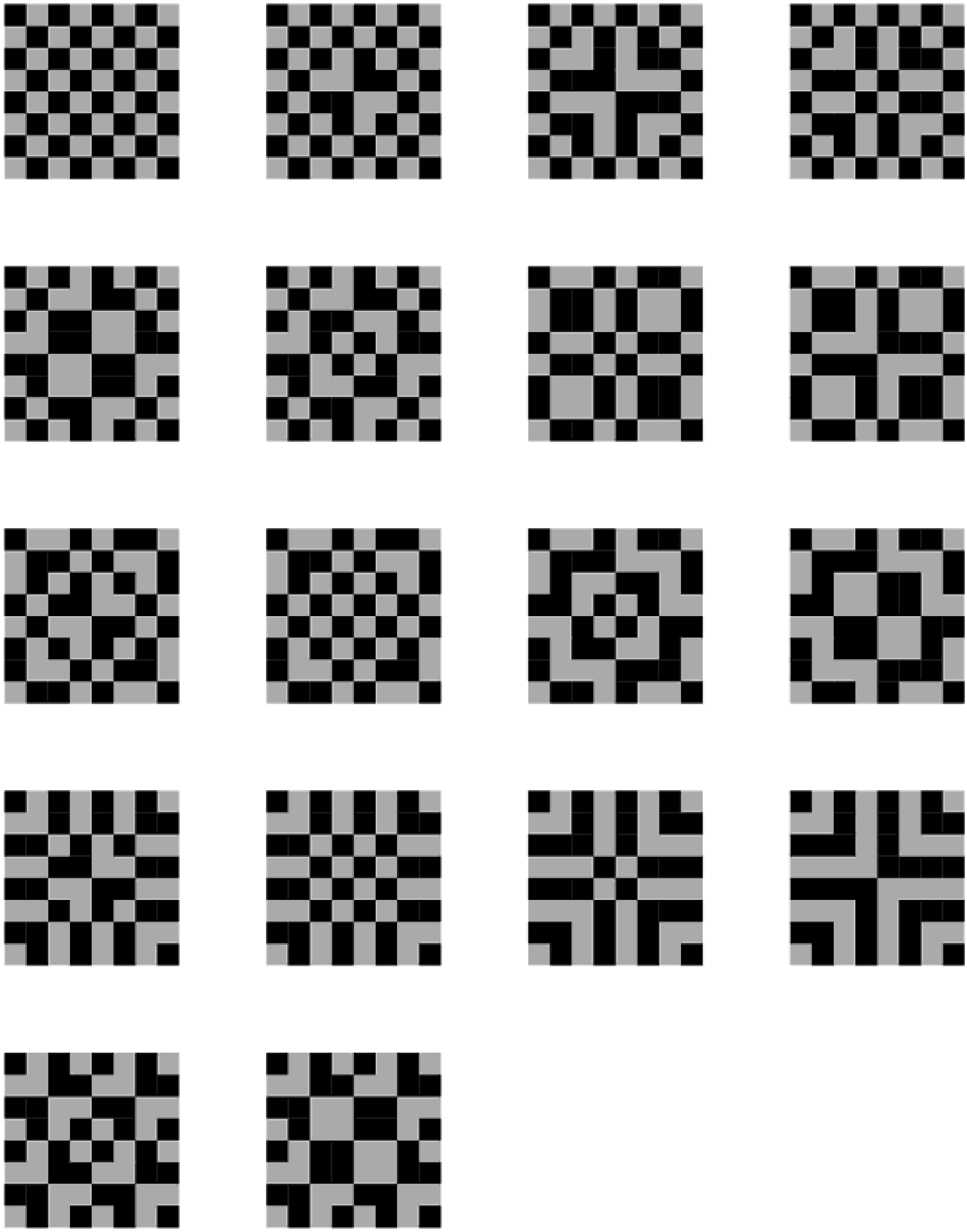


Figure 2.17

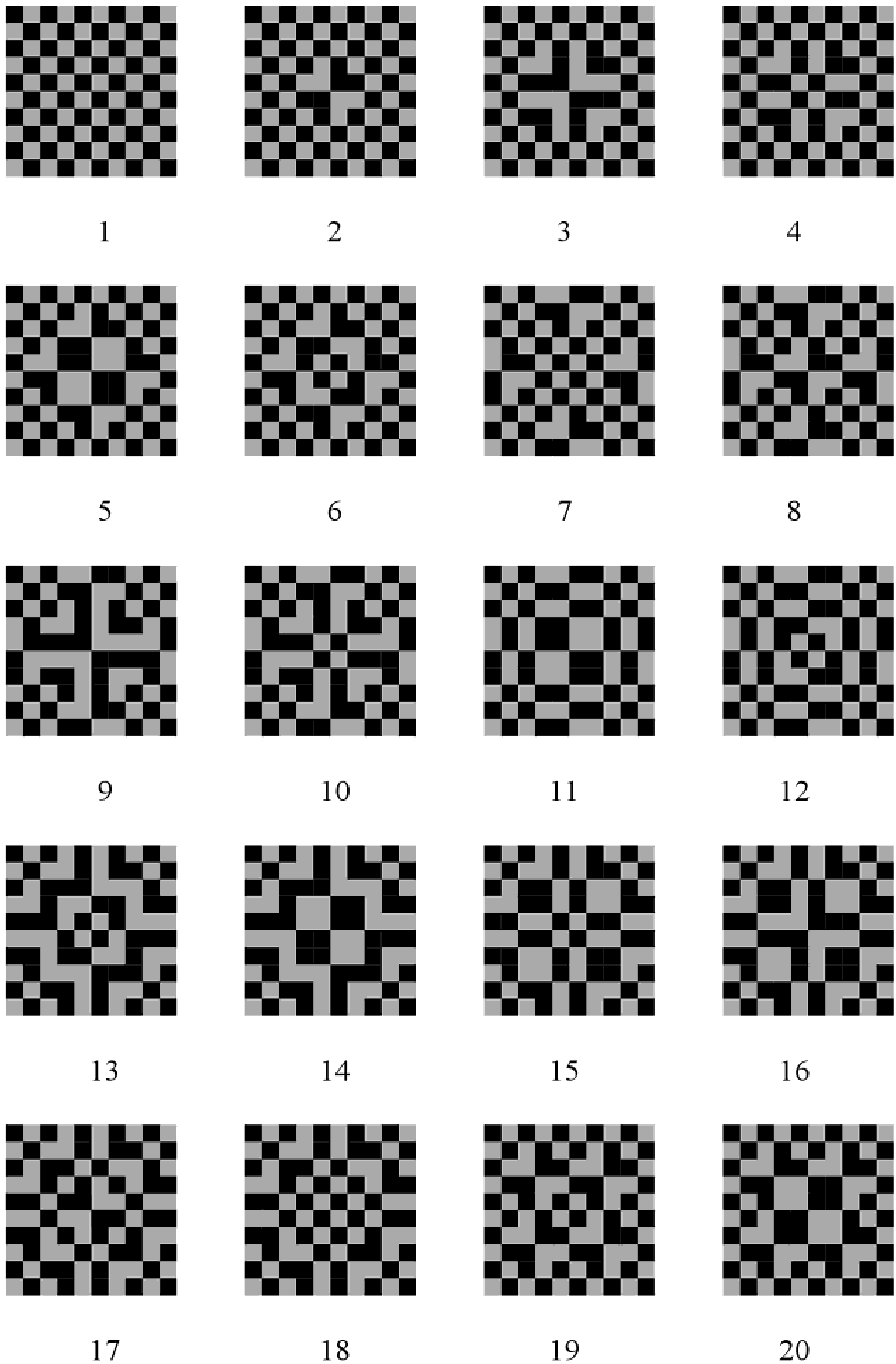


Figure 2.18 (First part)

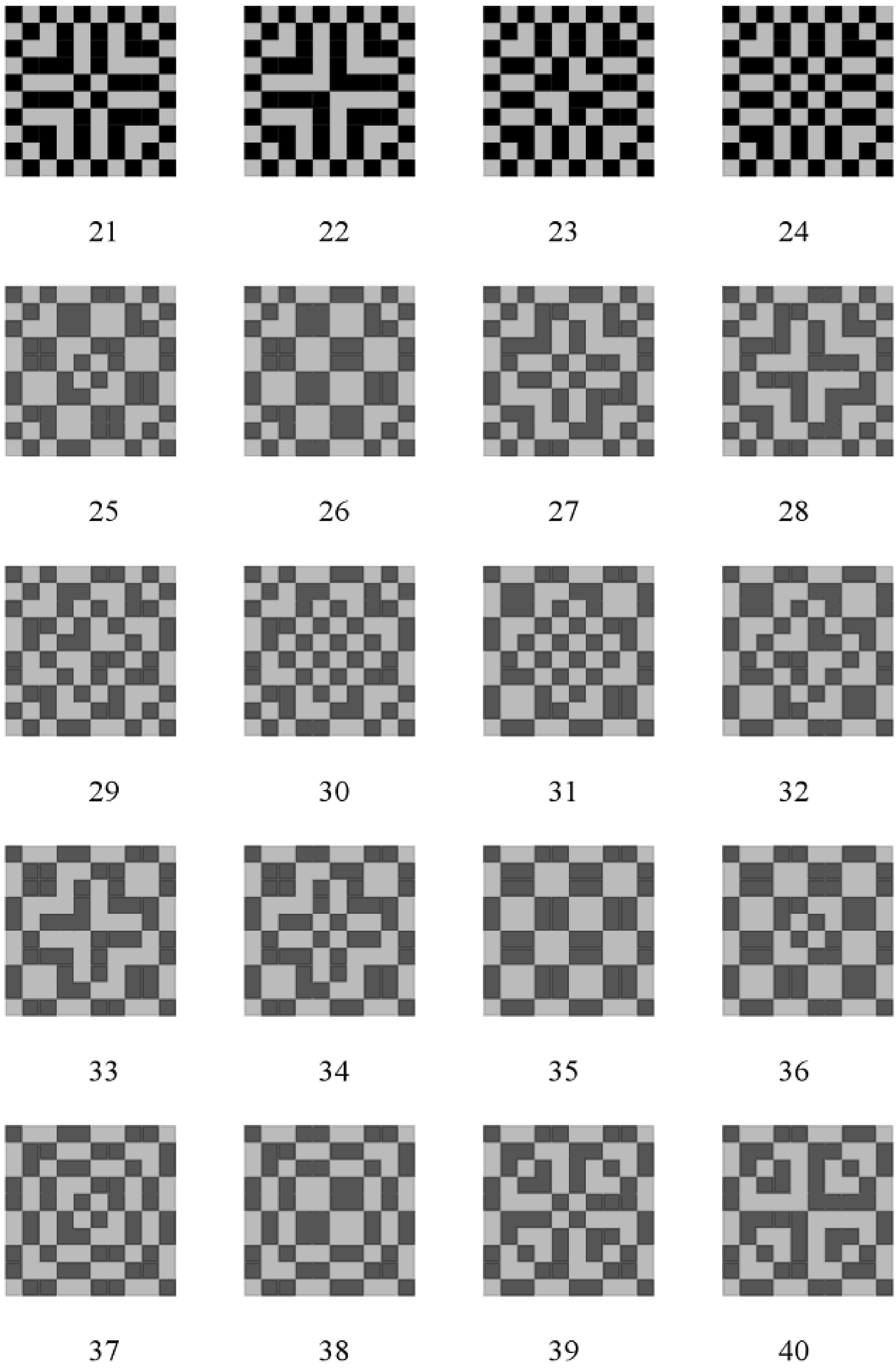
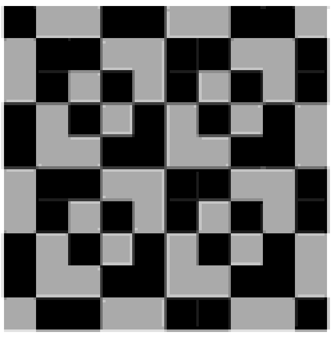


Figure 2.18 (Continued)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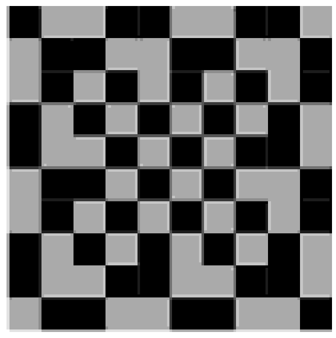
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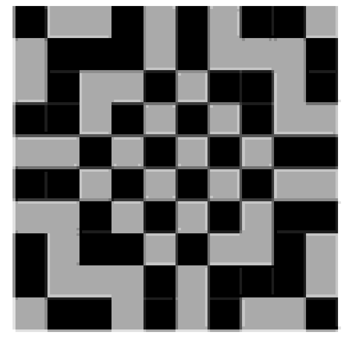
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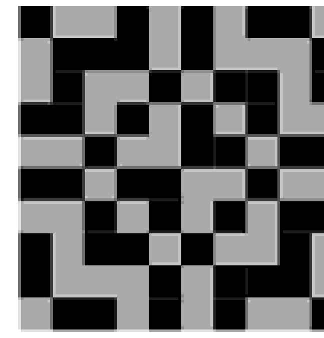
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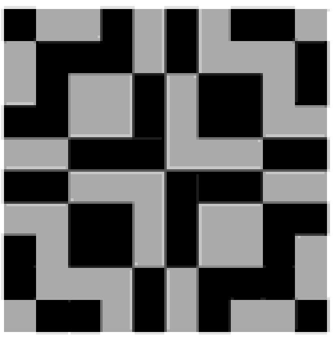
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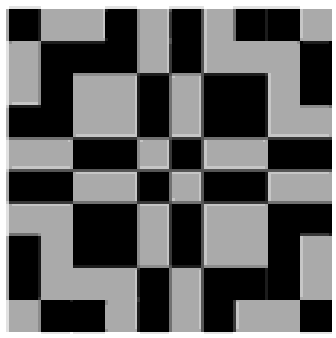
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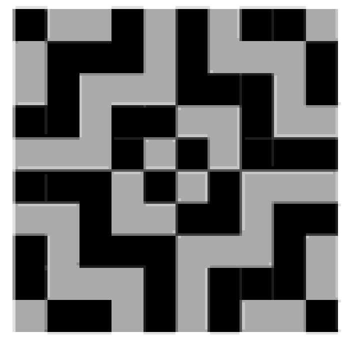
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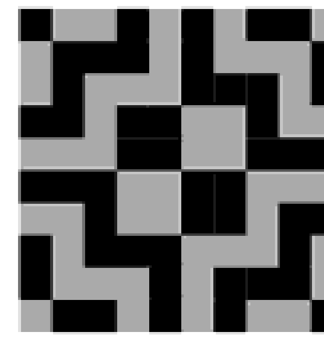
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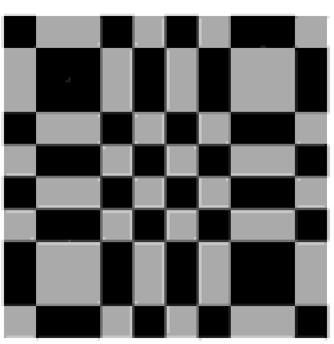
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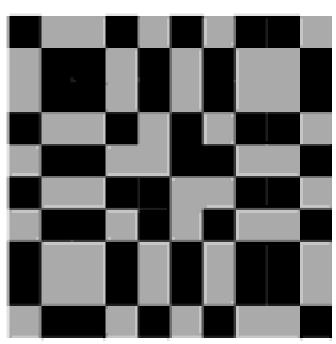
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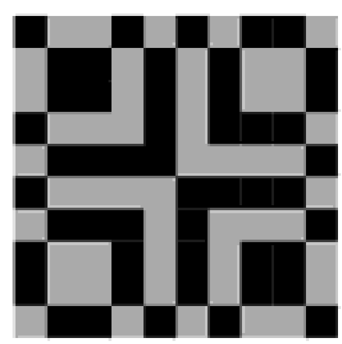
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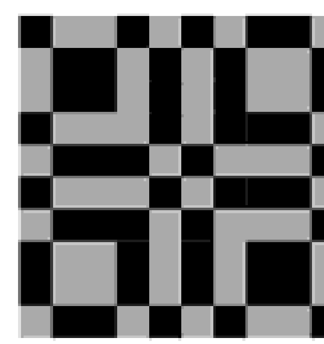
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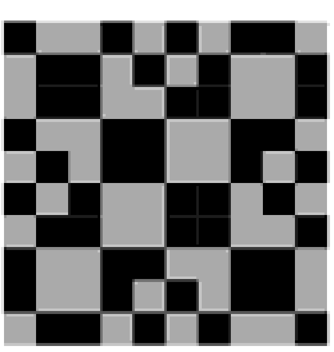
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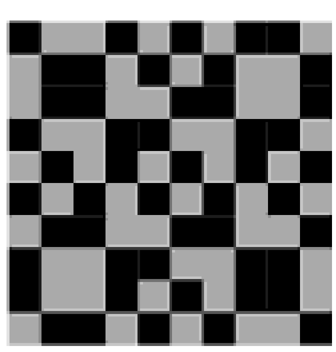
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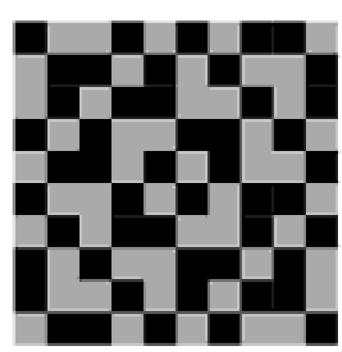
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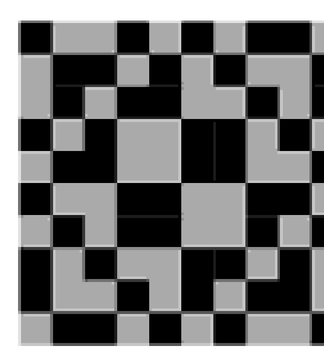
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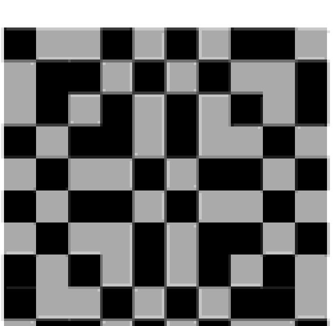
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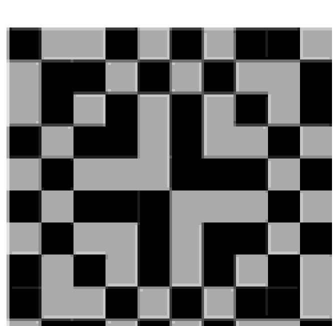
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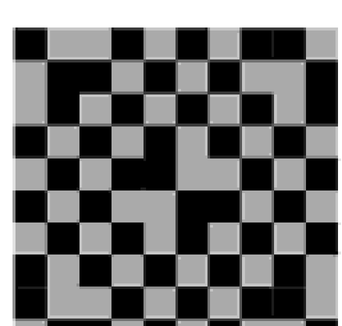
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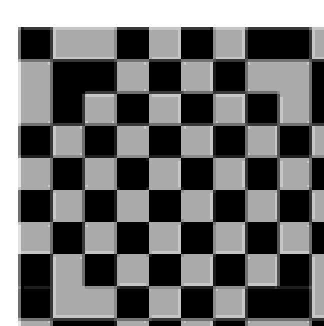
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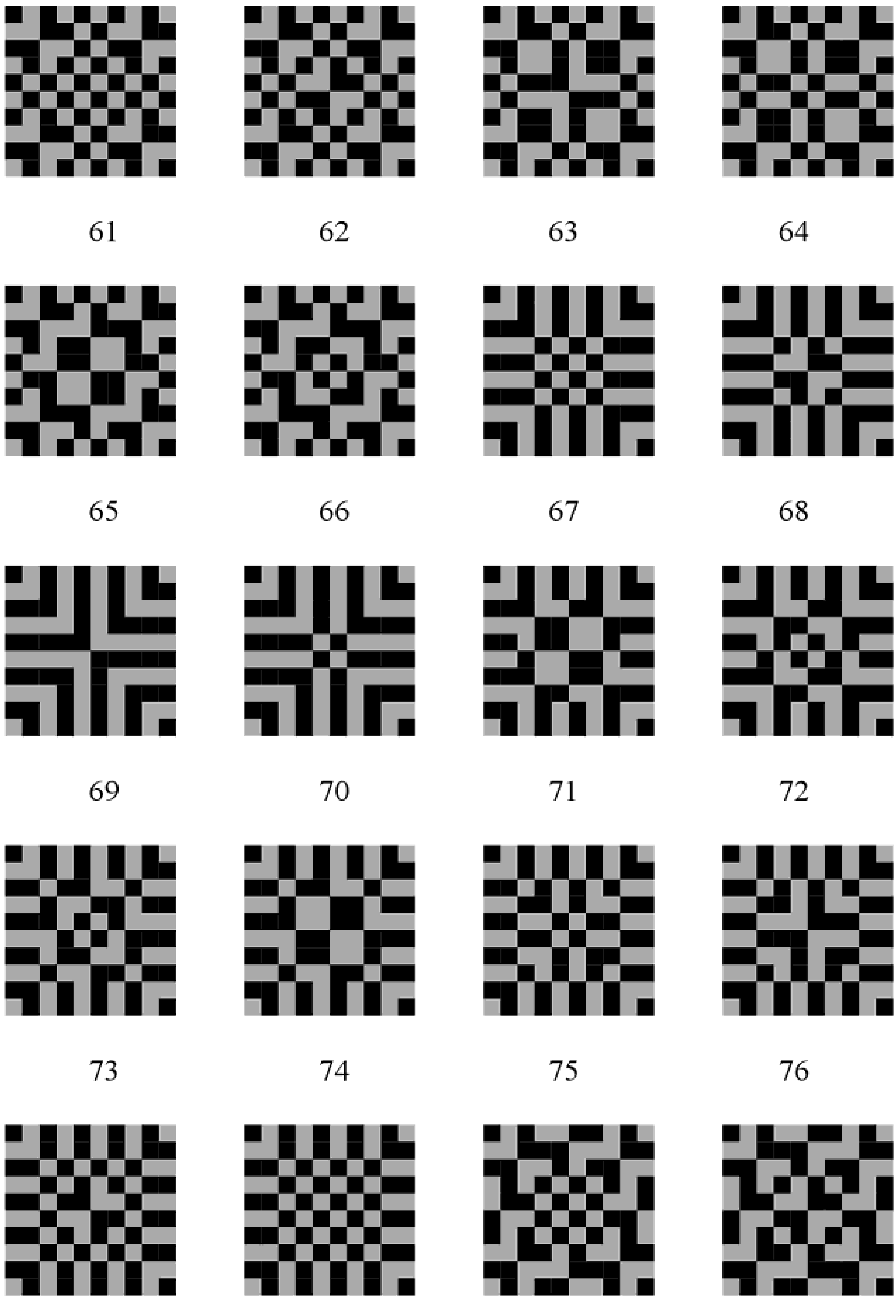
Figure 2.18 (continued)



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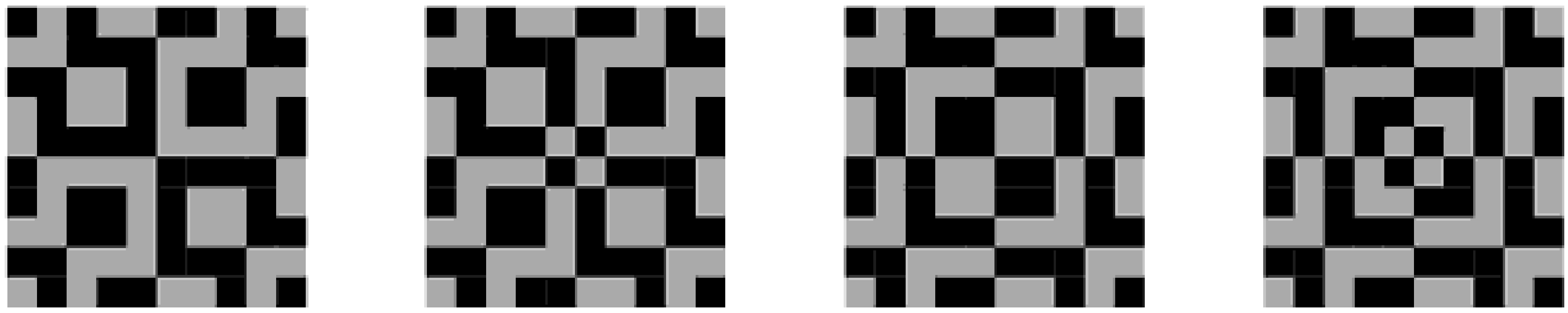
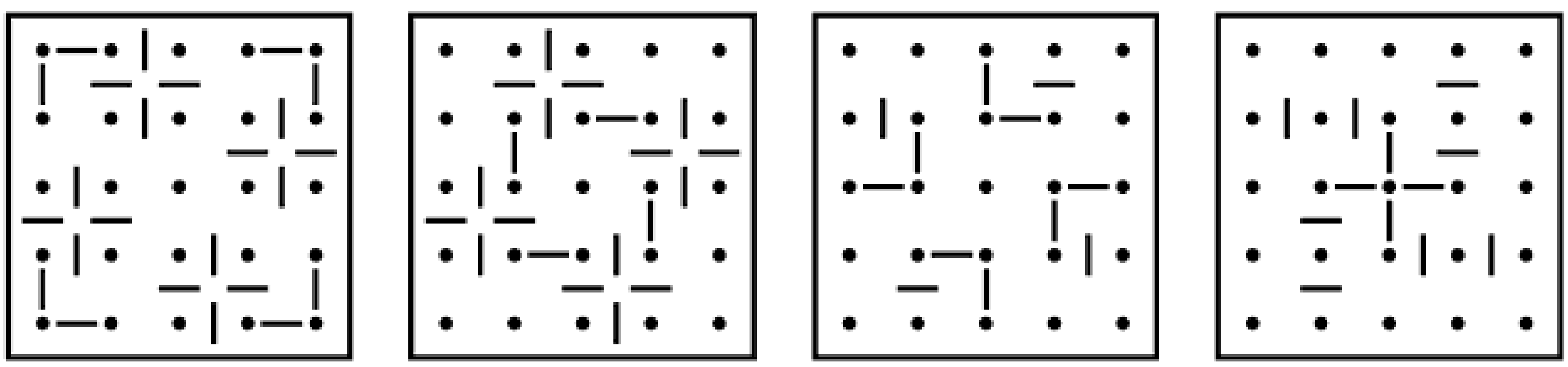


81 82 83 84

Figure 2.18 (Final part)



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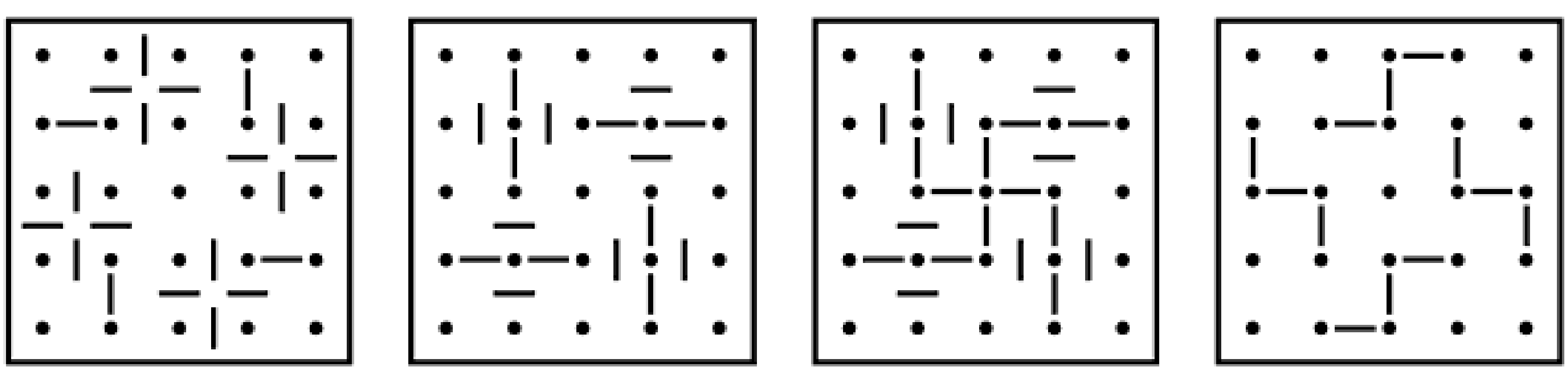

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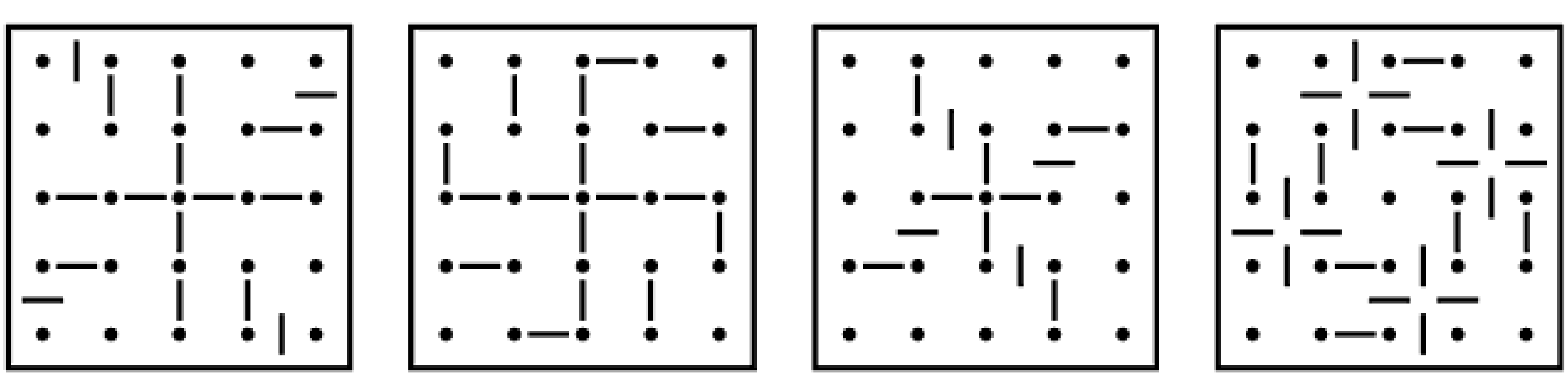

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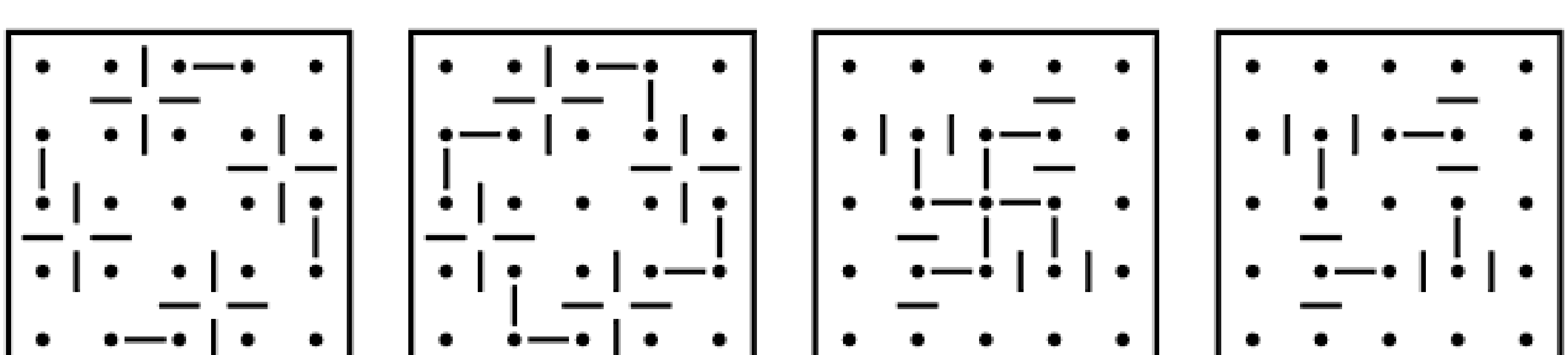
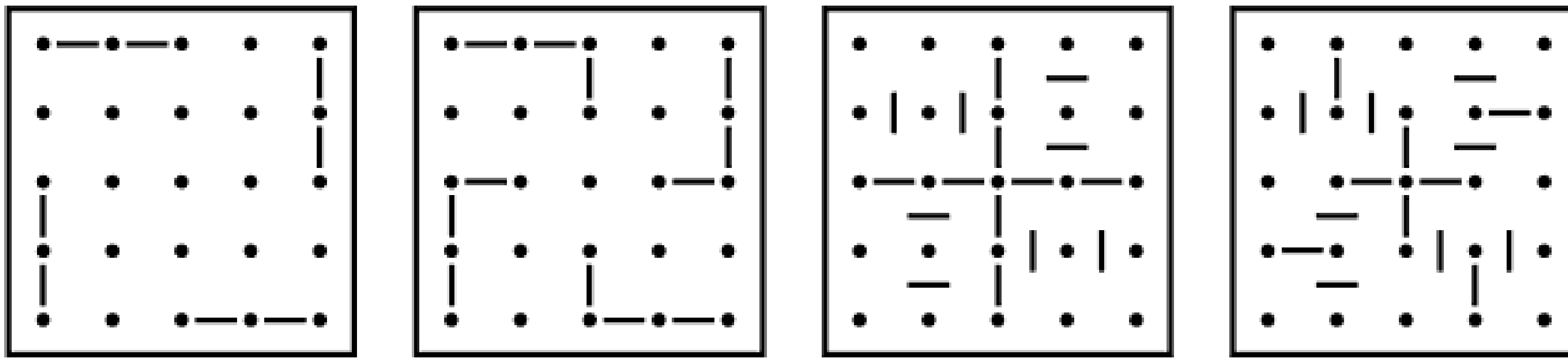
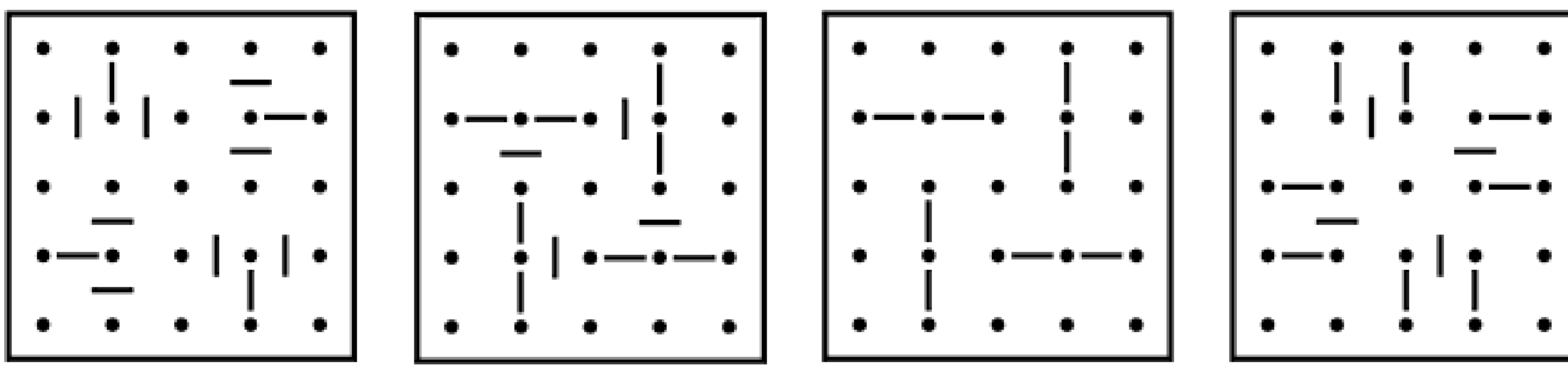


53 56 67 68

Figure 2.19 (First part)



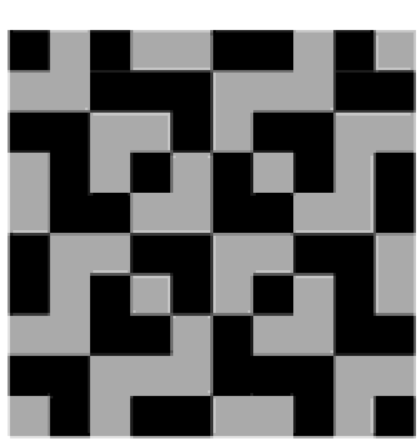
69 71 72 73



74 80 81 83

Figure 2.19 (Second part)

When we join four Lunda-designs of the type *d4'*, a new Lunda-design of the same type is obtained (see the example in Figure 2.20).



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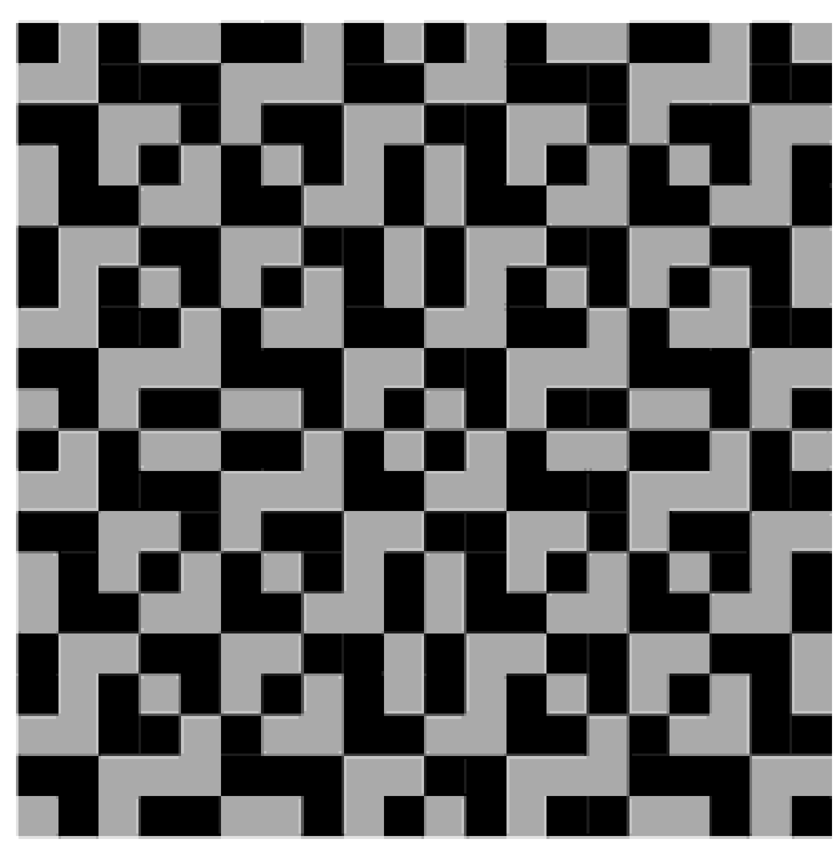


b

Figure 2.20

Figure 2.21 displays examples of 5x5 Lunda-designs — together with corresponding generating mirror designs — , which, although they do not have symmetry axes, do possess the property that a quarter-turn about the respective center reverses the colors, and consequently a half-turn preserves the colors (type *c4'*).

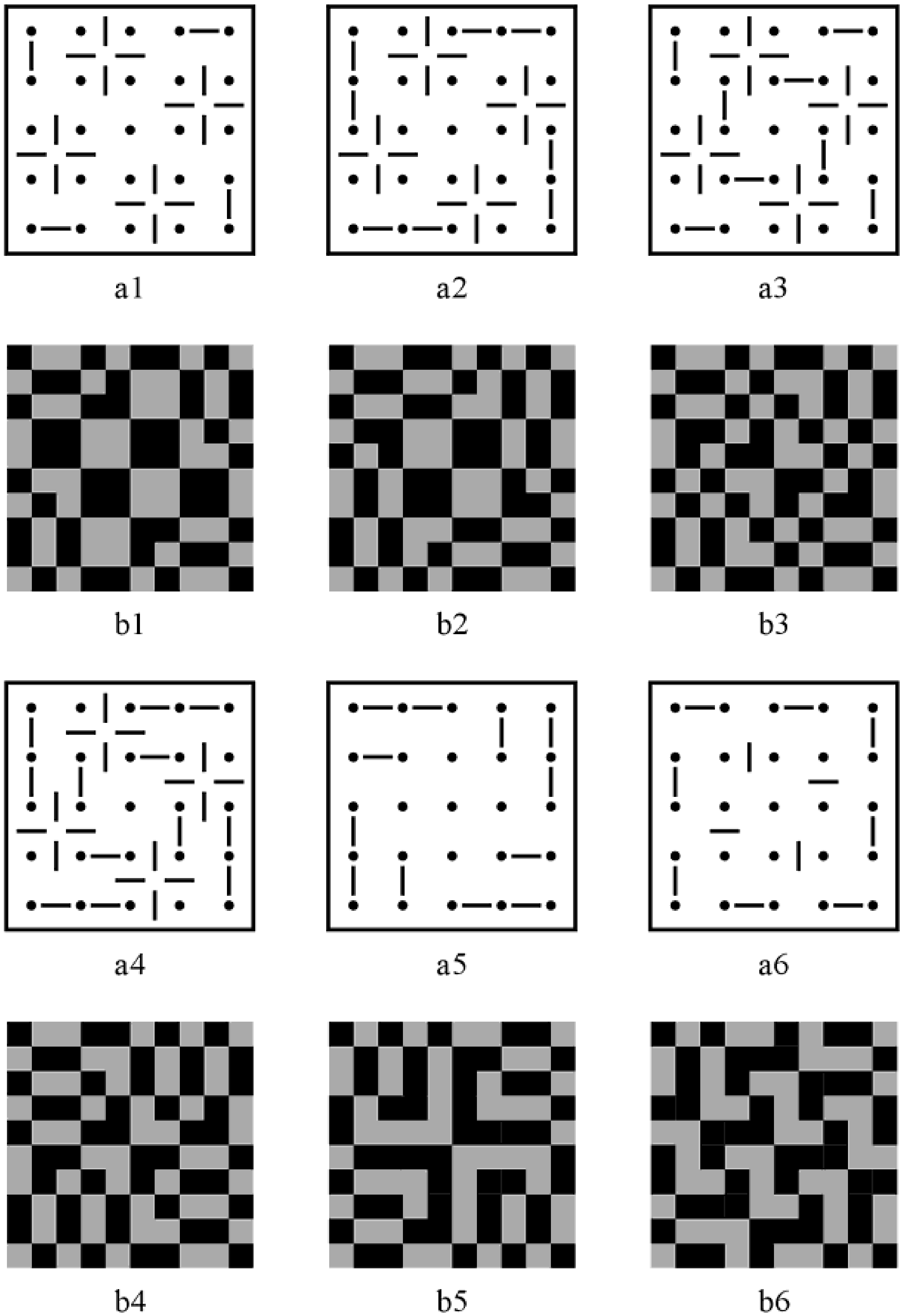


Figure 2.21 (First part)

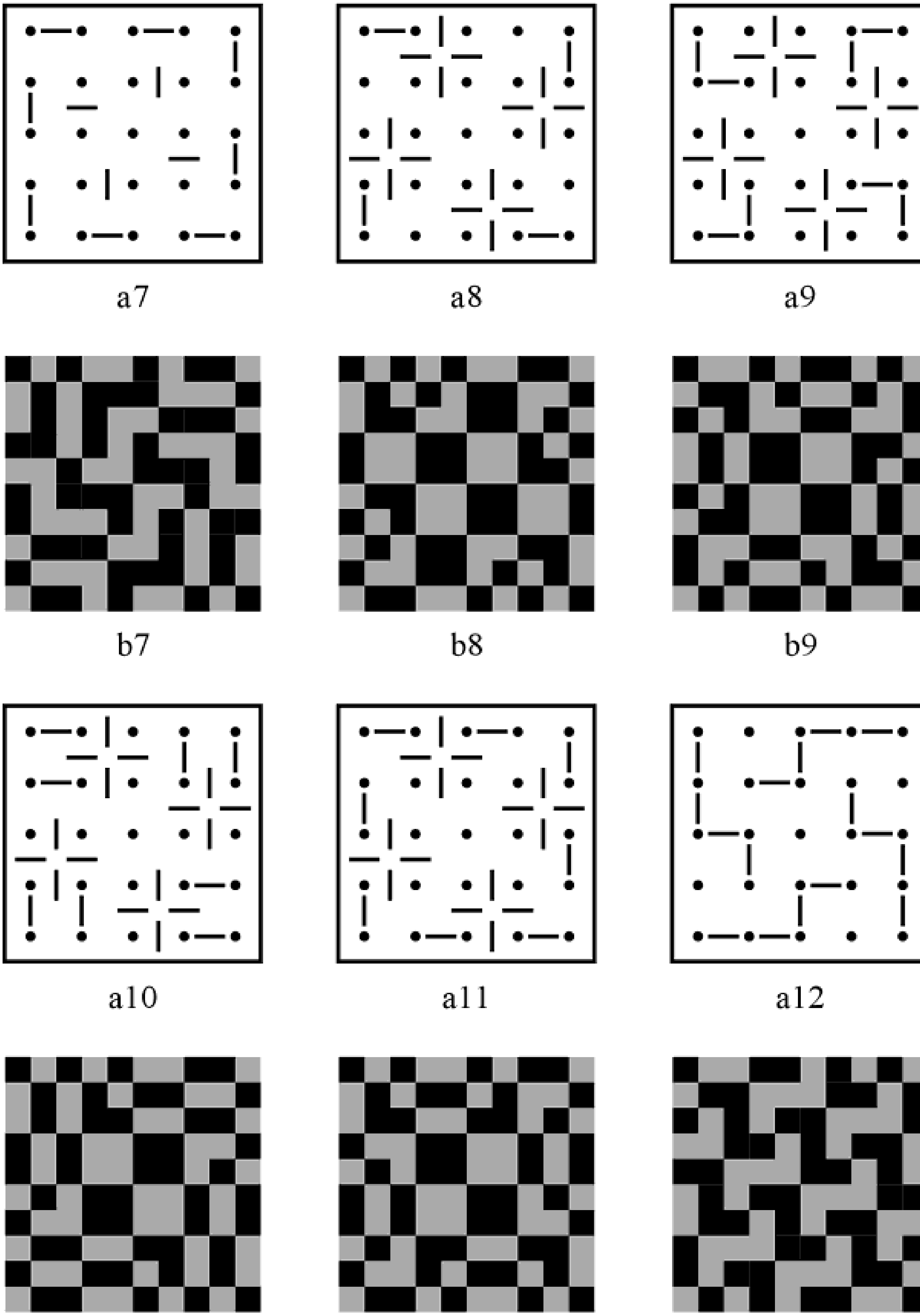
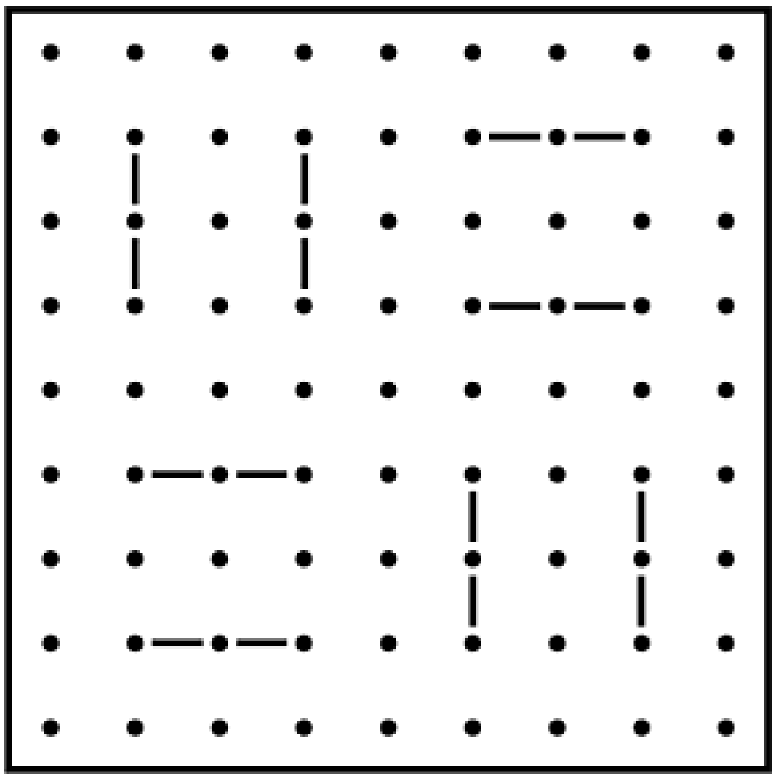
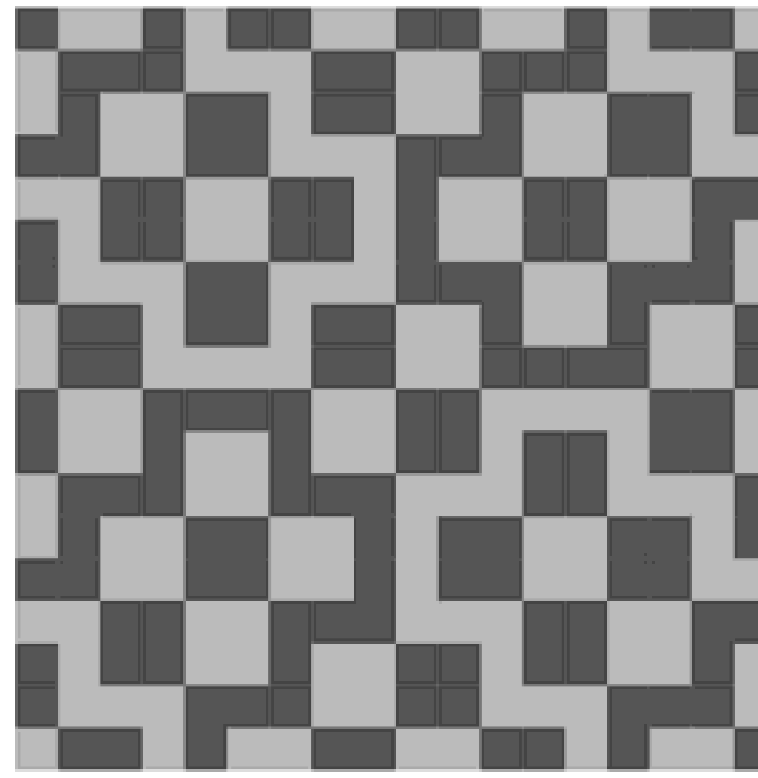


Figure 2.21 (second part)

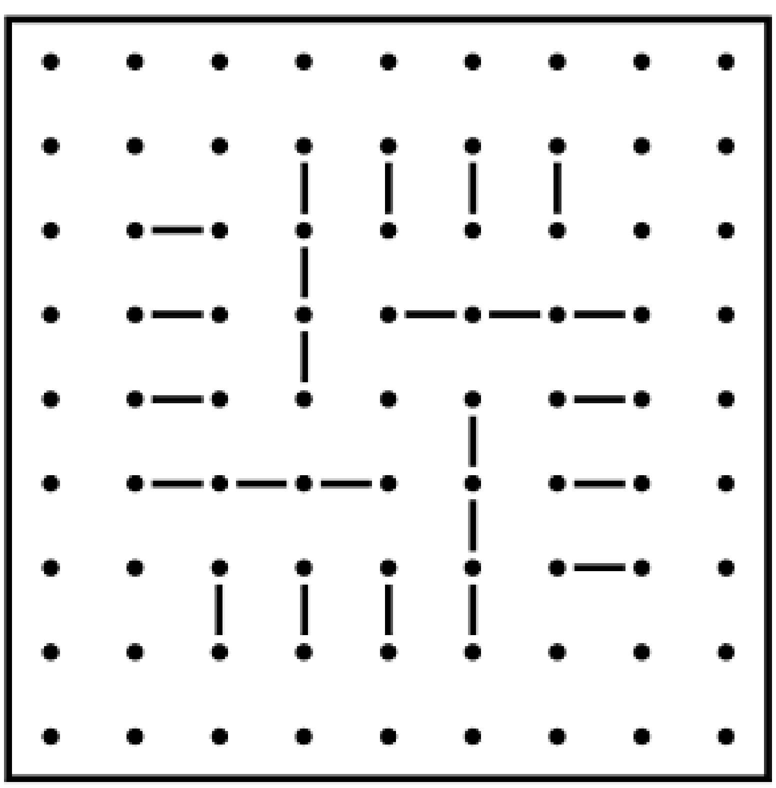
Figure 2.22 presents examples of 9x9 Lunda-designs of the type *d4'* together with corresponding generating mirror designs.



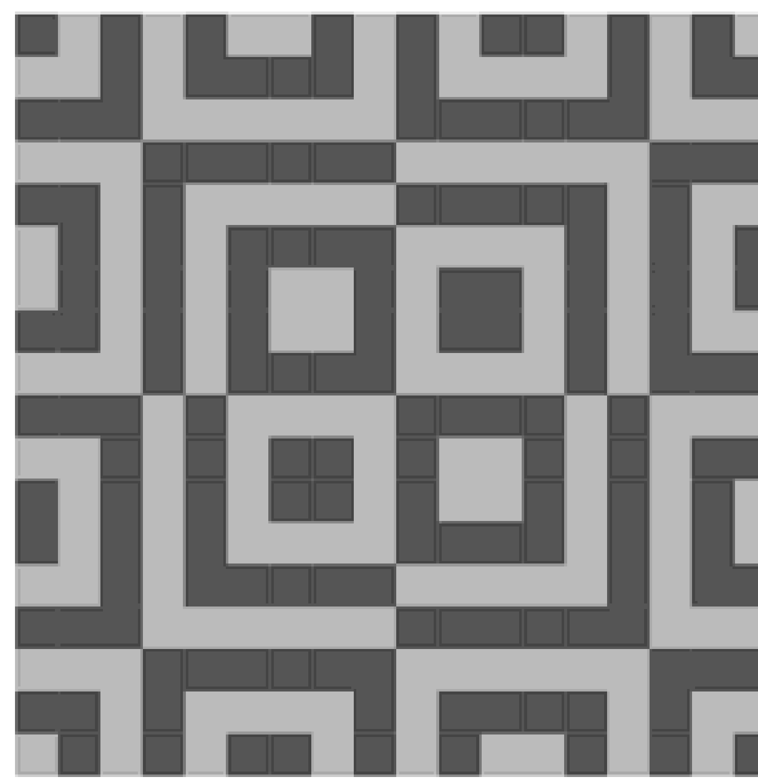
a1



b1



a2



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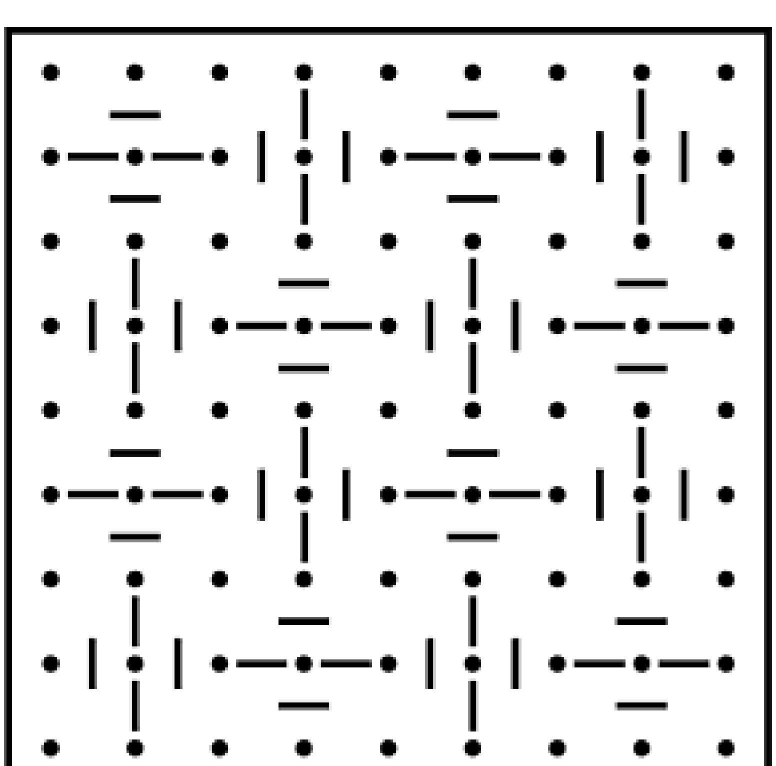

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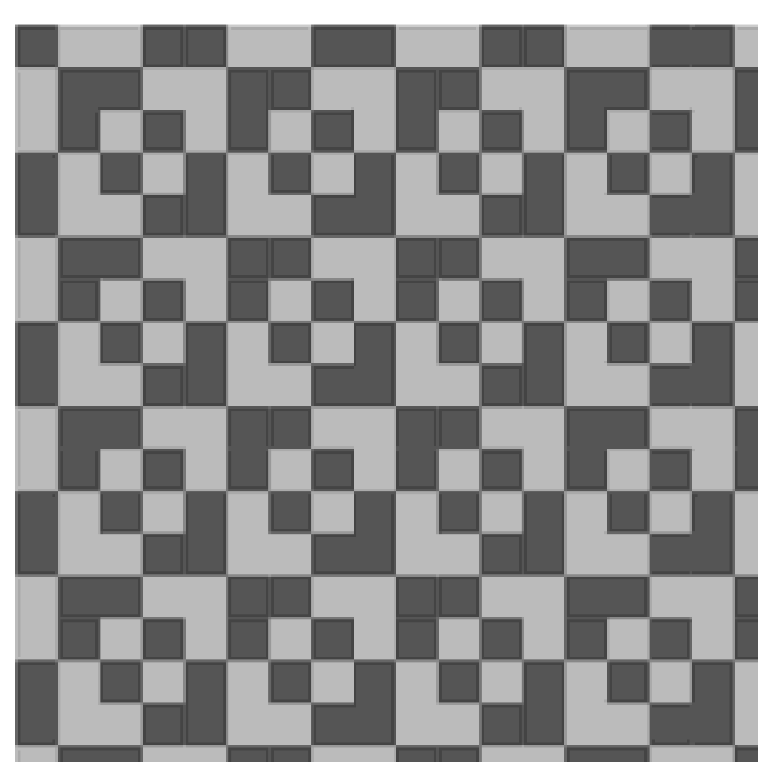
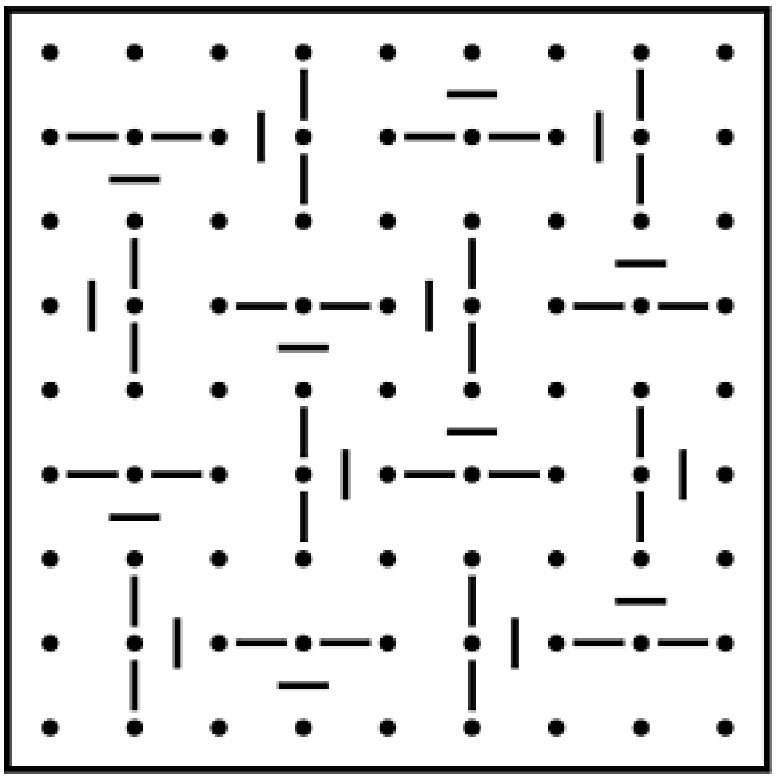
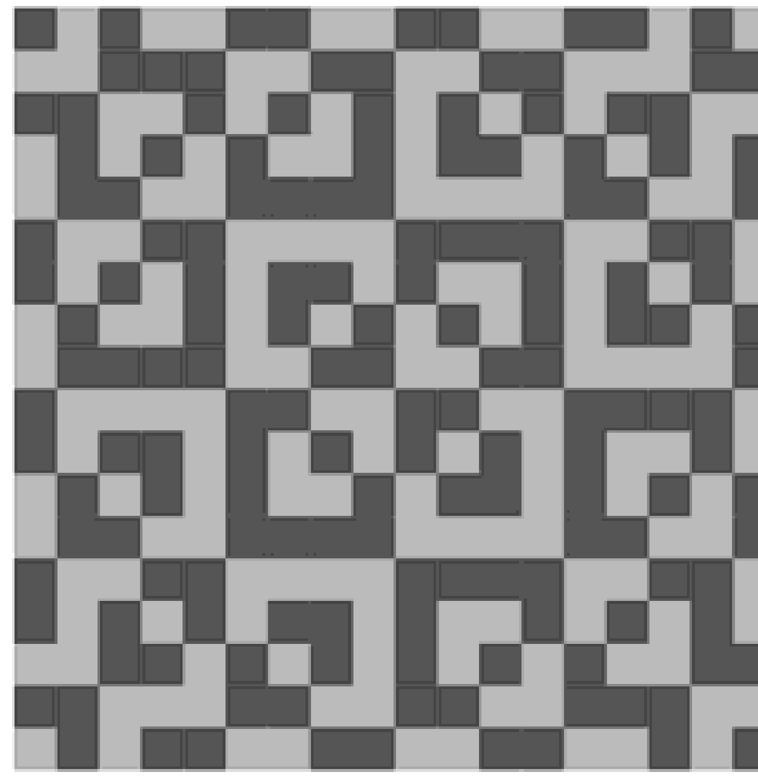


b3

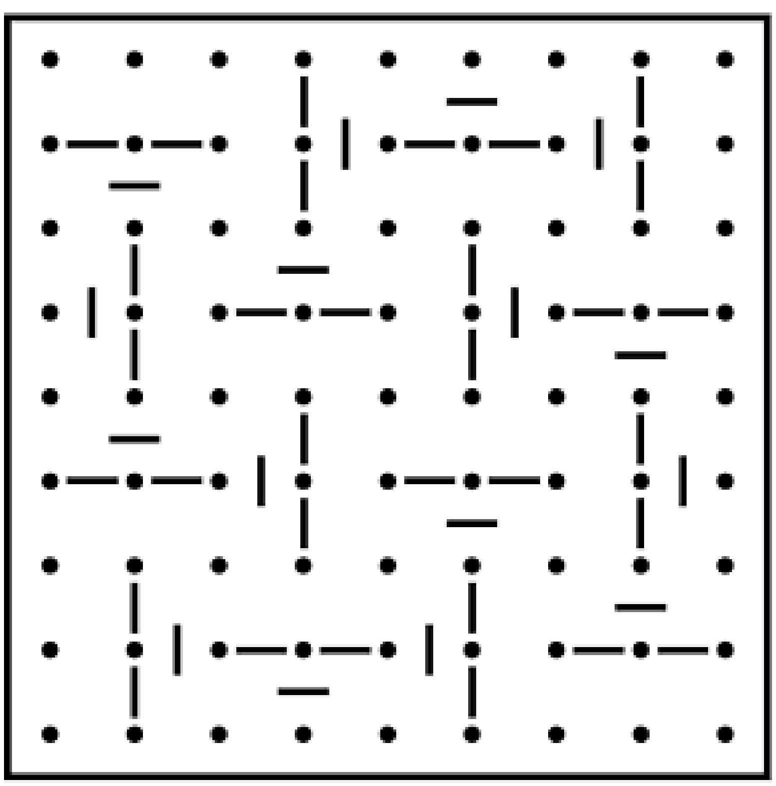
Figure 2.23 (first part)



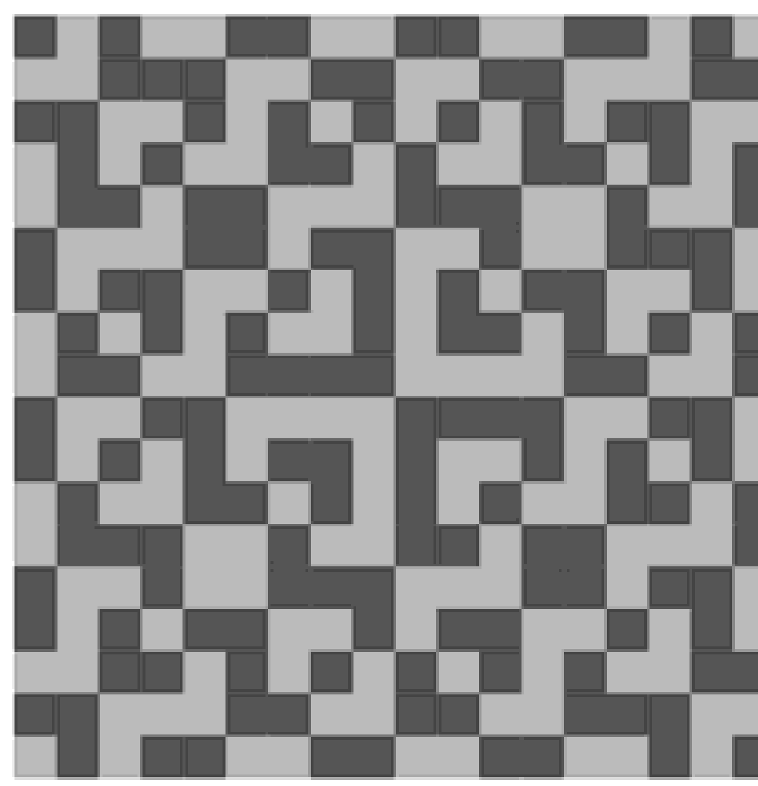
a4



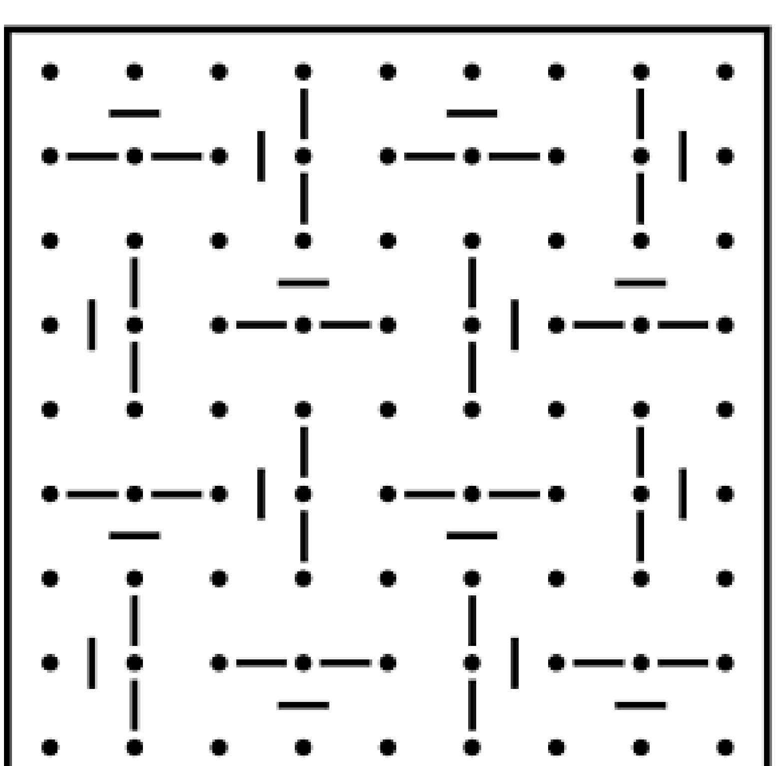
b4



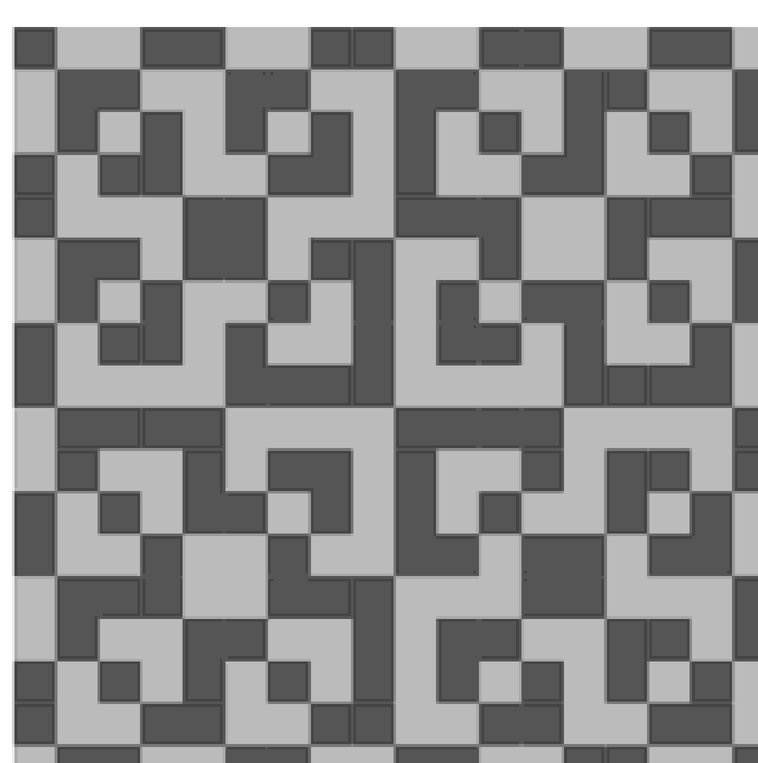
a5



b5



a6



b6

Figure 2.22 (second part)

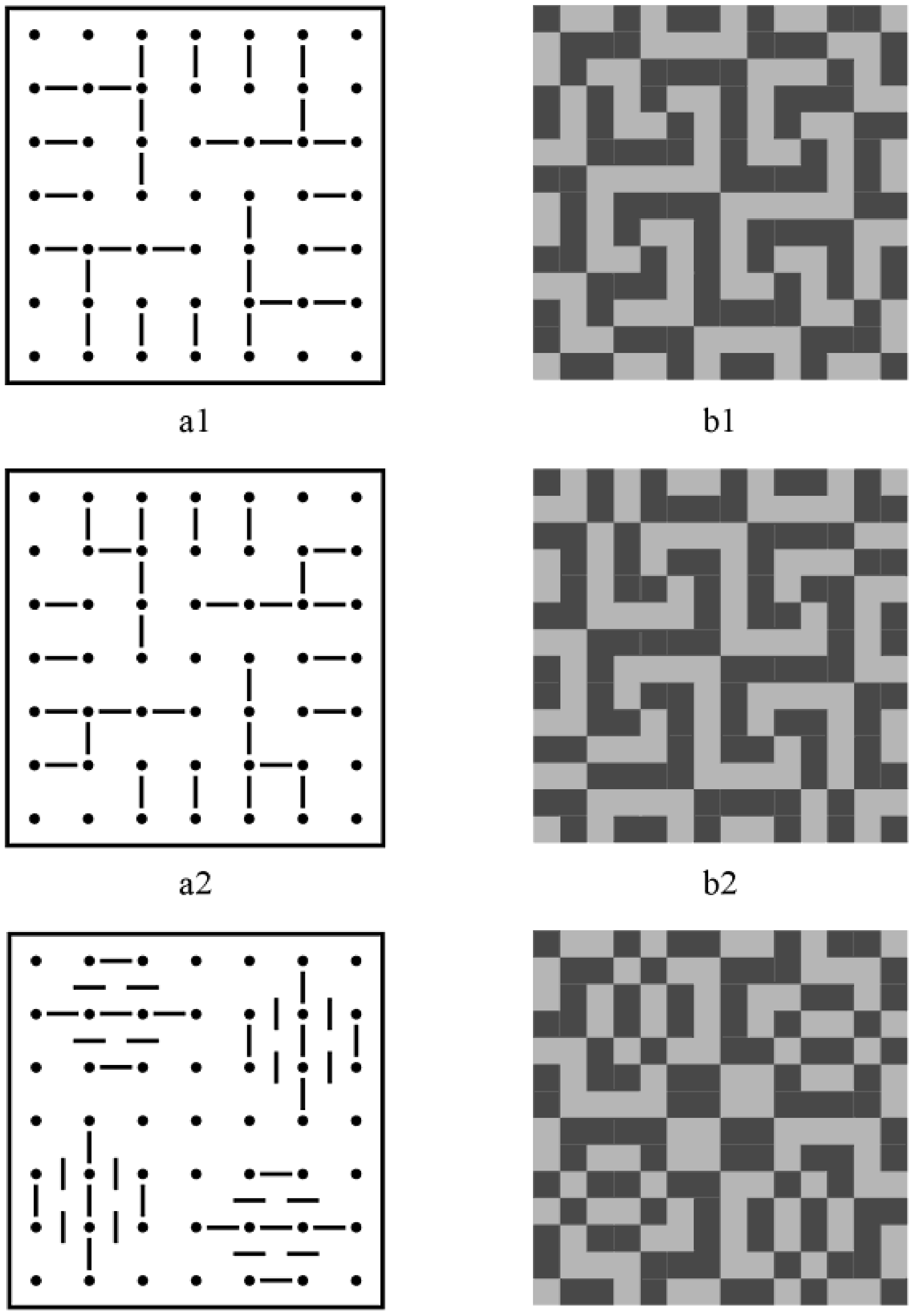


Figure 2.23

Figure 2.23 presents examples of 7x7 Lunda-designs of the type *c4'*. Examples of 9x9 and 13x13 Lunda-designs of the same type *c4'* are displayed in Figures 2.24 and 2.25. Two examples of 9x9 Lunda-designs, which admit reflections in their diagonals are given in Figure 2.26.

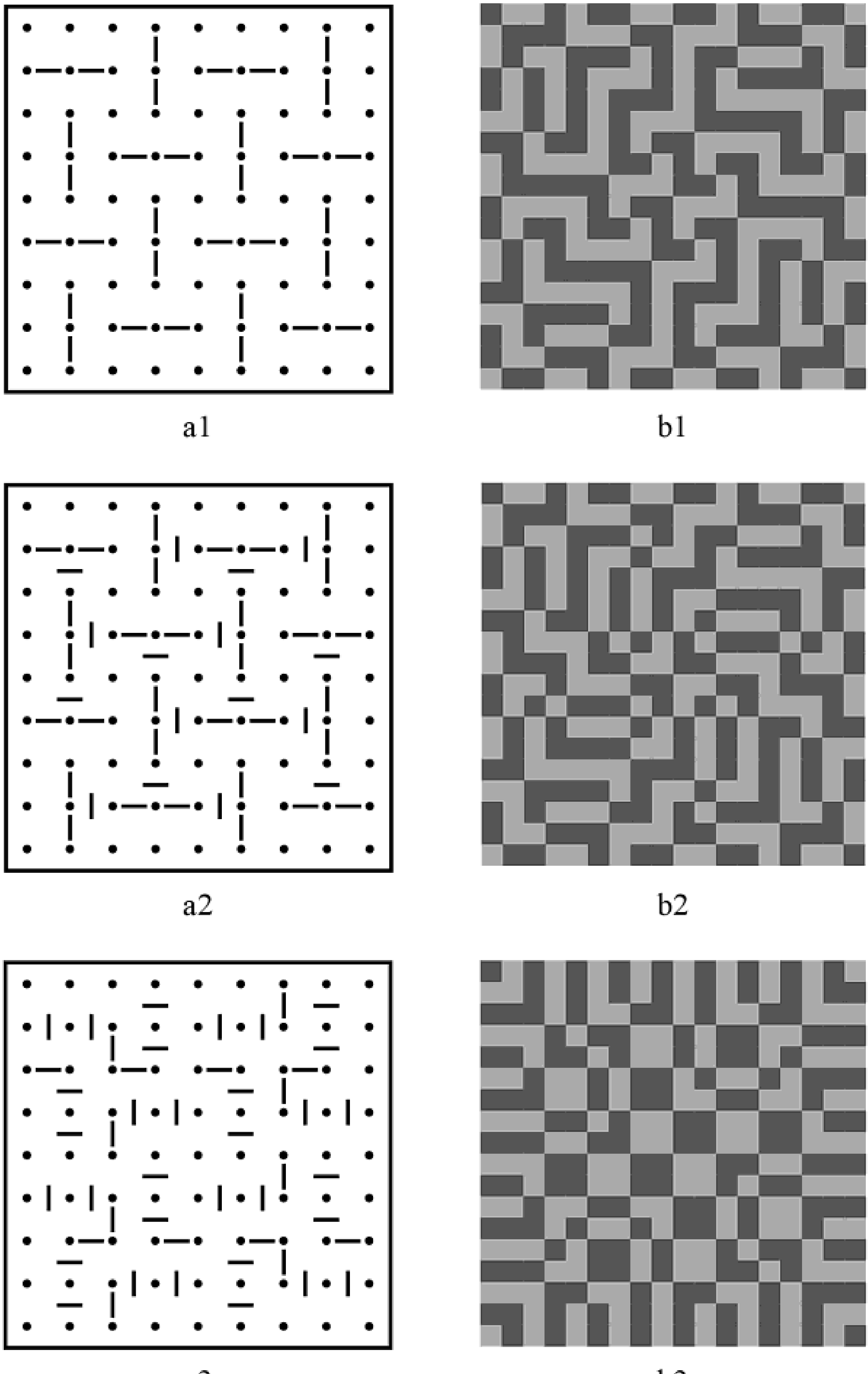
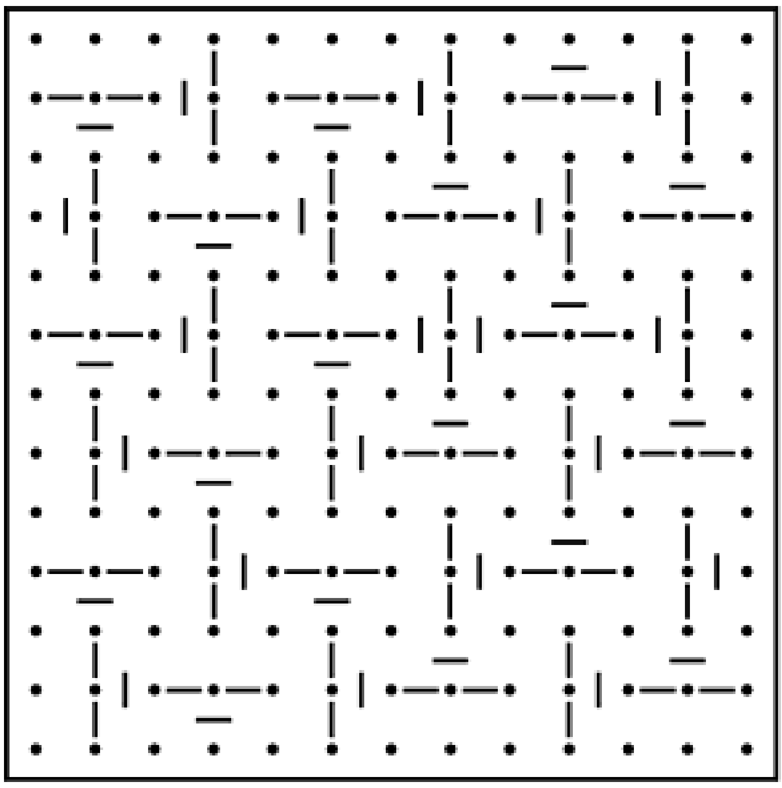
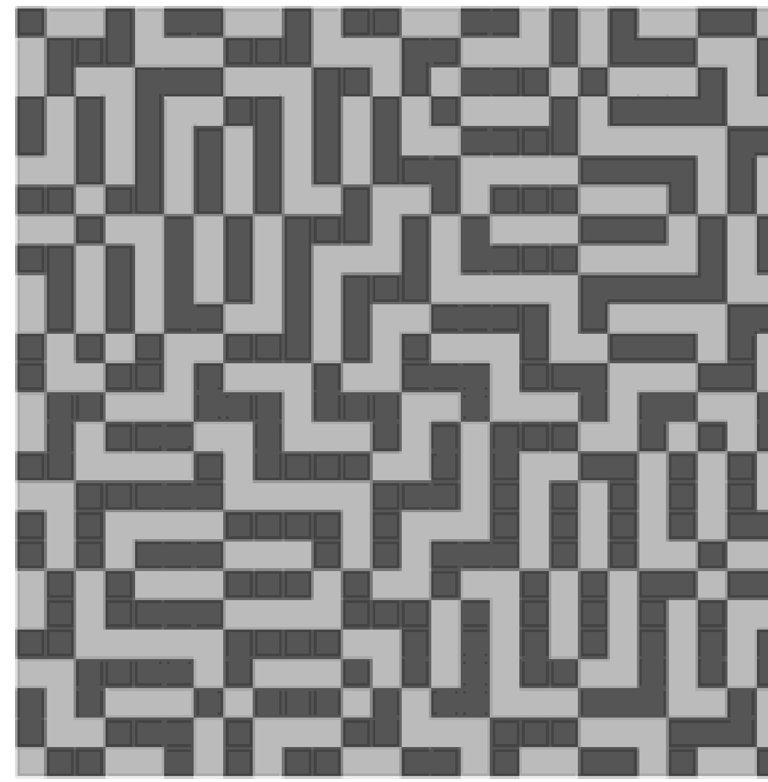


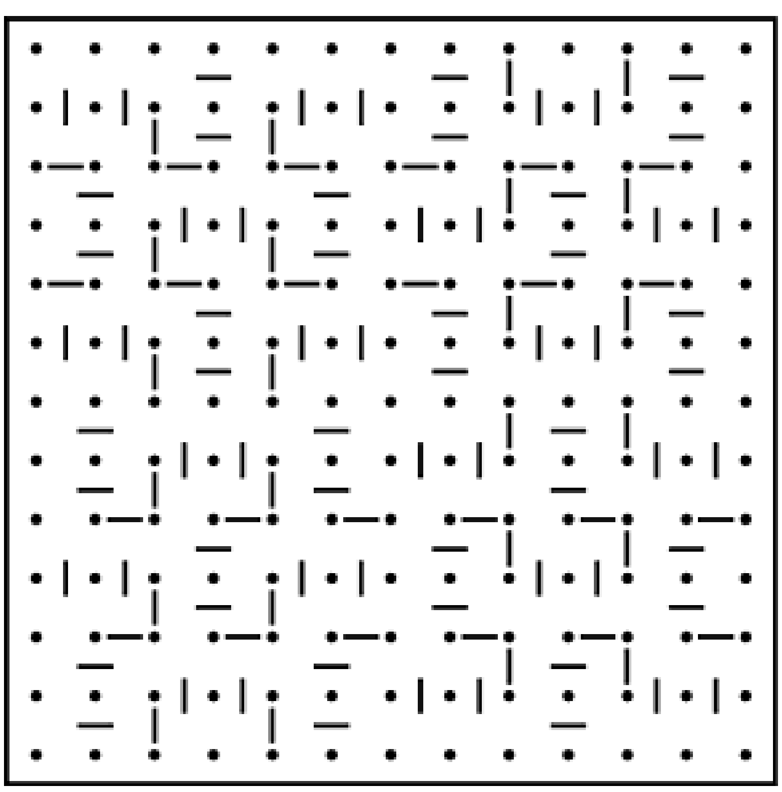
Figure 2.24



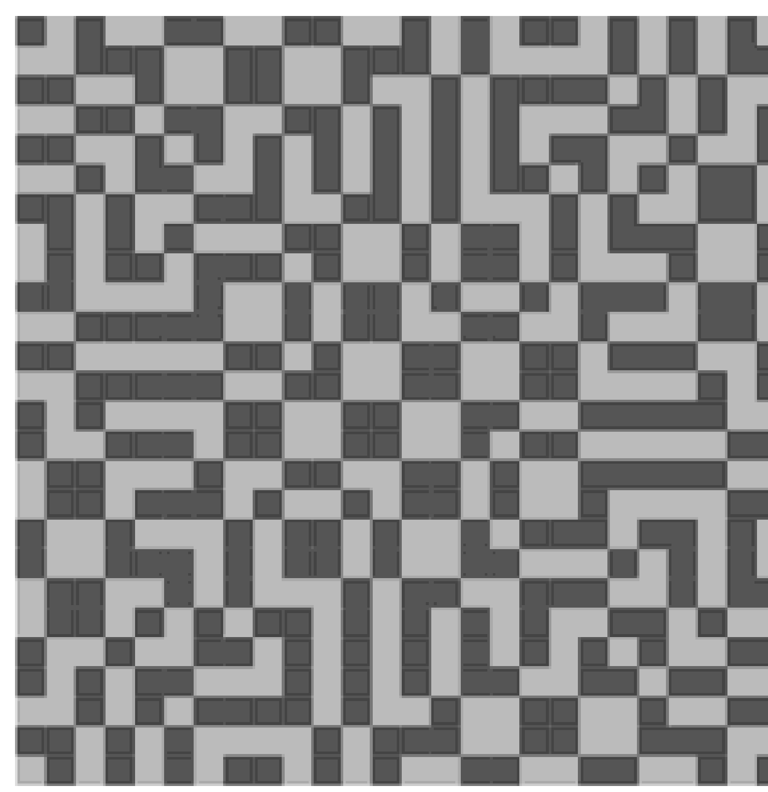
a1



b1

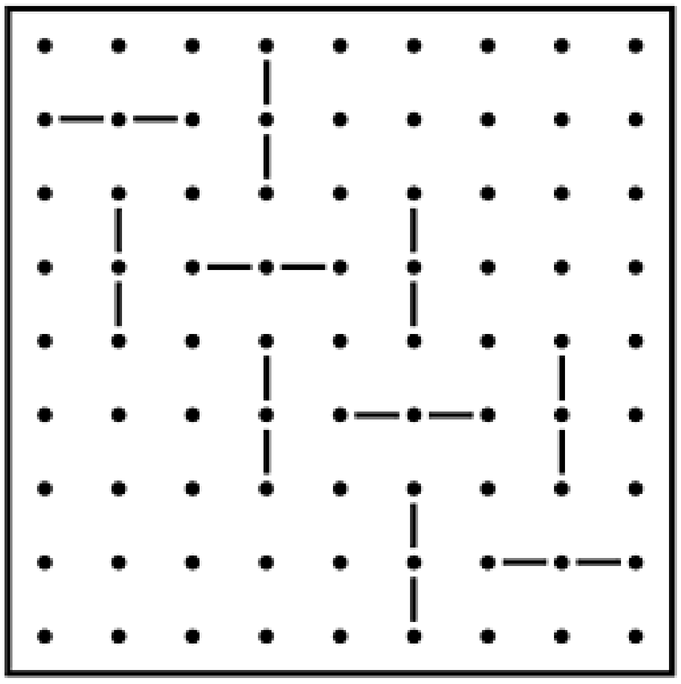


a2

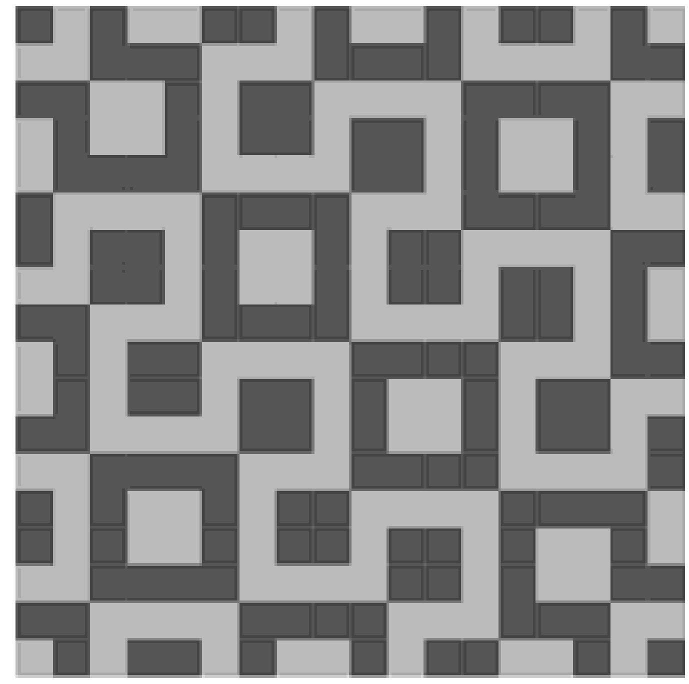


b2

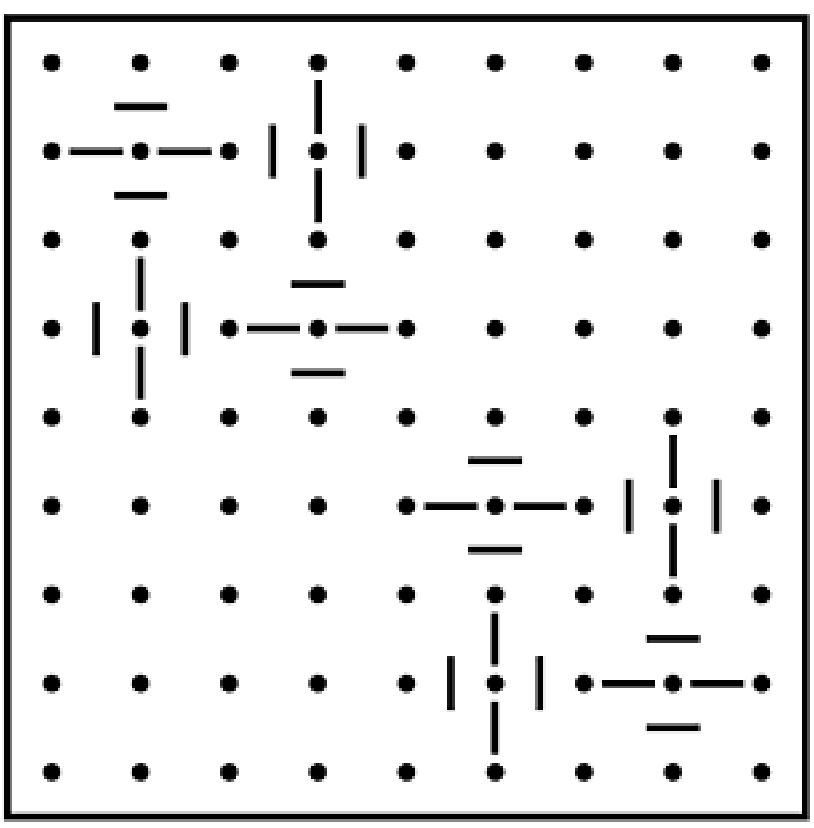
Figure 2.25



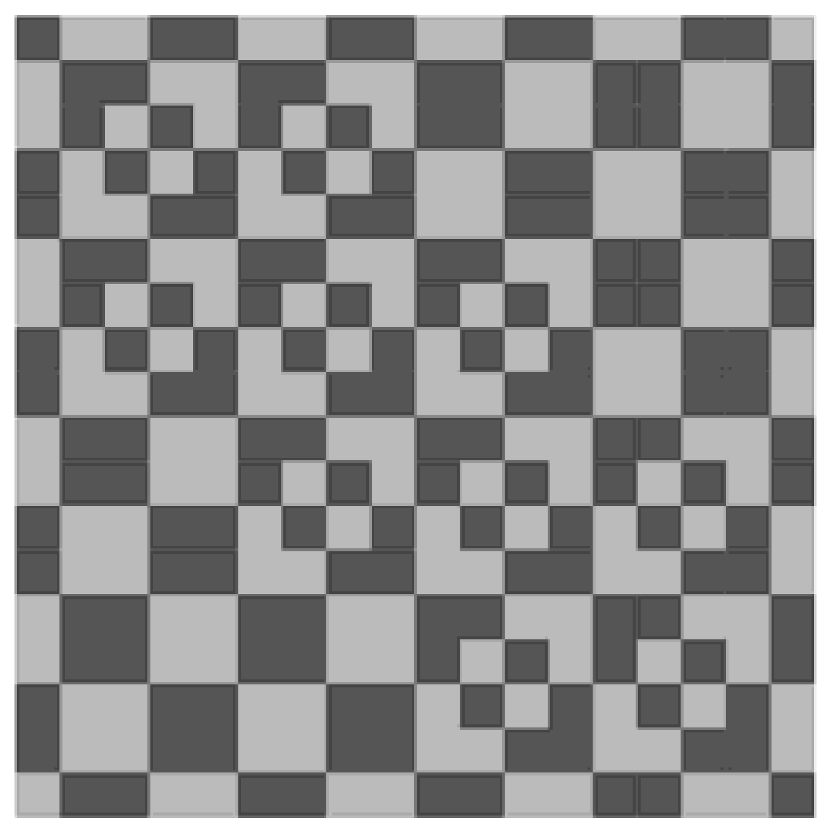
a1



b1



a2



b2

Figure 2.26

2.6 Square Lunda-designs and fractals

Square Lunda-designs may be used to build up fractals, that is geometrical figures with a built in self-similarity (see e.g. Lauwerier). Figure 2.27 shows the first three phases of building up a fractal on the base of a 2x2 Lunda-design of the type *d4'*. The fractal itself admits only two reflections. Figure 2.28 presents the first two phases of the construction of a fractal on the base of a 4x4 Lunda-design.

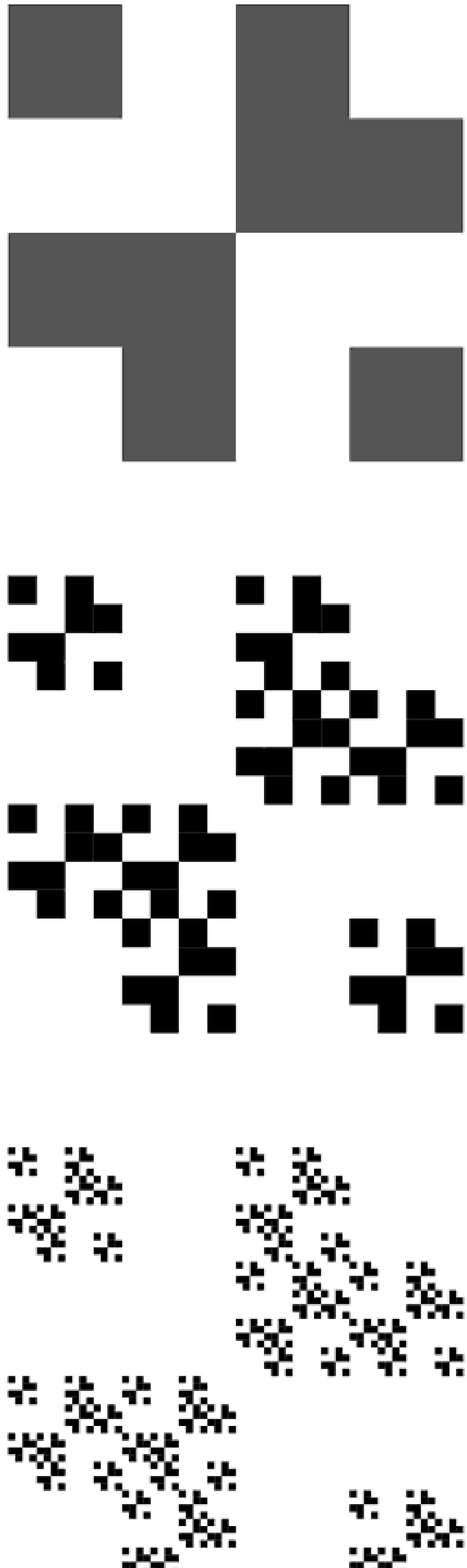


Figure 2.27

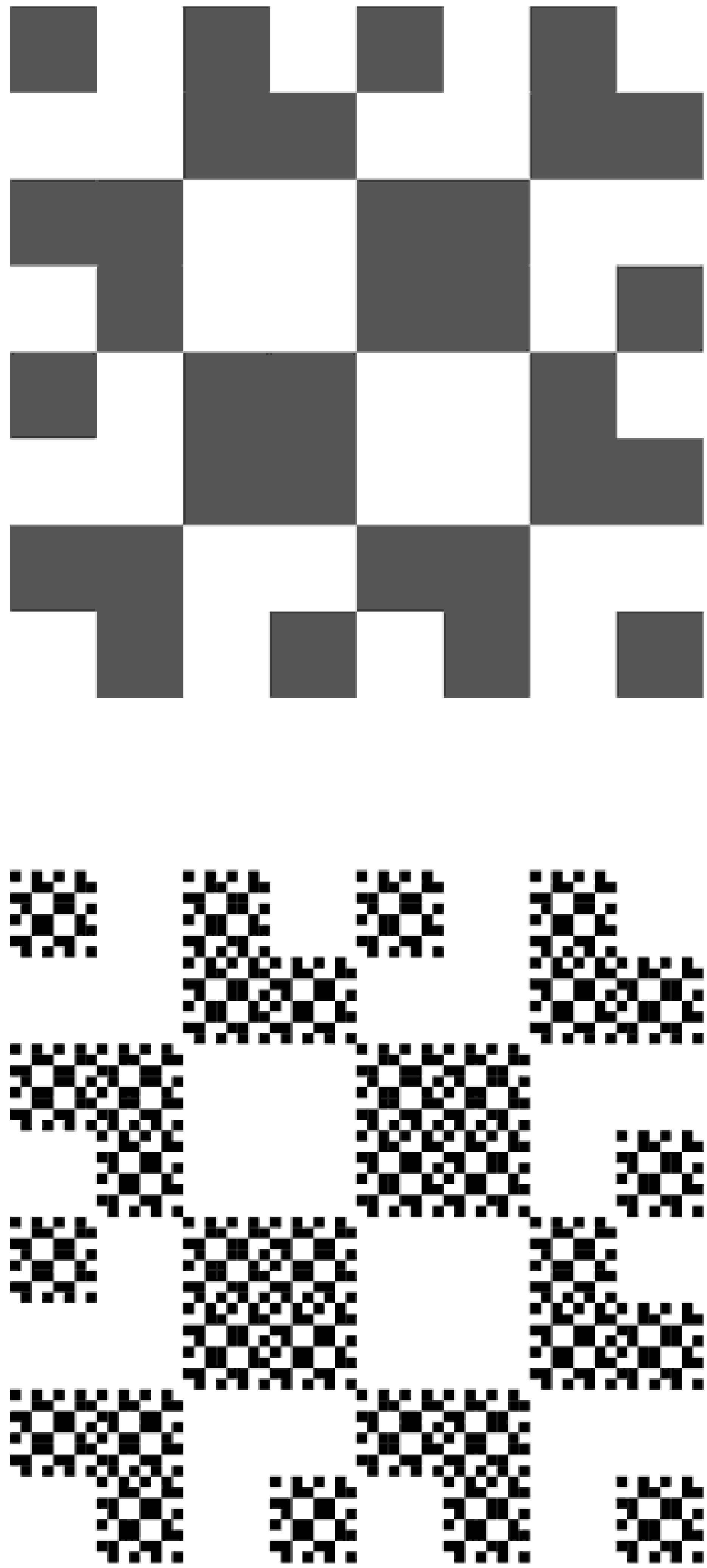


Figure 2.28

2.7 Generalization of Lunda-designs

As Lunda-designs may be considered as matrices, it is quite natural to define addition of Lunda-designs in terms of matrix addition: the sum of two (or more) matrices (of the same dimensions) is the matrix in which the elements are obtained by adding corresponding elements (see the example in Figure 2.29).

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&0&1&0&0&1&1&0\\\hline 0&1&0&1&1&0&0&1\\\hline 1&0&0&1&0&1&1&0\\\hline 0&1&1&0&1&0&0&1\\\hline\end{array} + \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&0&1&0&1&0&1&0\\\hline 0&0&1&1&0&1&0&1\\\hline 1&1&0&0&0&1&1&0\\\hline 0&1&0&1&1&0&0&1\\\hline\end{array}$$

$$=$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 2&0&2&0&1&1&2&0\\\hline 0&1&1&2&1&1&0&2\\\hline 2&1&0&1&0&2&2&0\\\hline 0&2&1&1&2&0&0&2\\\hline\end{array}$$

or

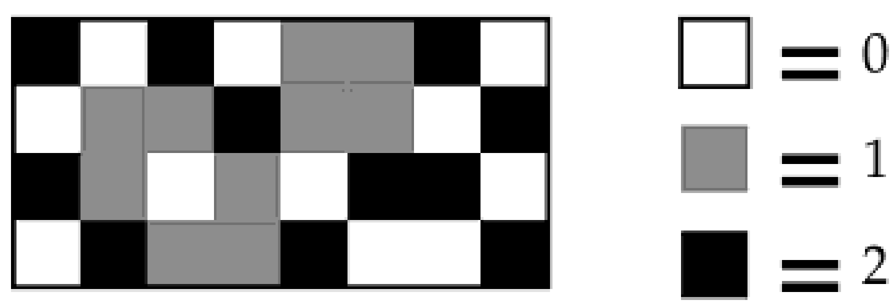


Figure 2.29

The sum of k mxn Lunda-designs may be called an **mxn Lunda-k-design**. The Lunda-k-designs inherit the following symmetry properties:

(i) The sum of the elements in any row is equal to km;
(ii) The sum of the elements in any column is equal to kn;
(iii) The sum of the integers in two border unit squares of any grid point in the first or last rows or columns is equal to k;
(iv) The sum of the integers in the four unit squares between two arbitrary (vertical or horizontal) neighbor grid points is always $2k$. [1]

[1] If we define a Lunda-design not as a (0,1) - matrix, but as a (-1,1) - matrix, these properties assume the following expressions:
(i) The sum of the elements in any row is equal to 0;
(ii) The sum of the elements in any column is equal to 0;
(iii) The sum of the integers in two border unit squares of any grid point in the first or last rows or columns is equal to 0;
(iv) The sum of the integers in the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points is always 0.

Once again, properties (i) and (ii) guarantee a global equilibrium for each row and column. Properties (iii) and (iv) guarantee more local equilibriums.

The characteristics (i), (ii), (iii), and (iv) may be used to define Lunda-k-designs of dimensions mxn. The characteristics (iii) and (iv) are sufficient for this definition, as they imply (i) and (ii).

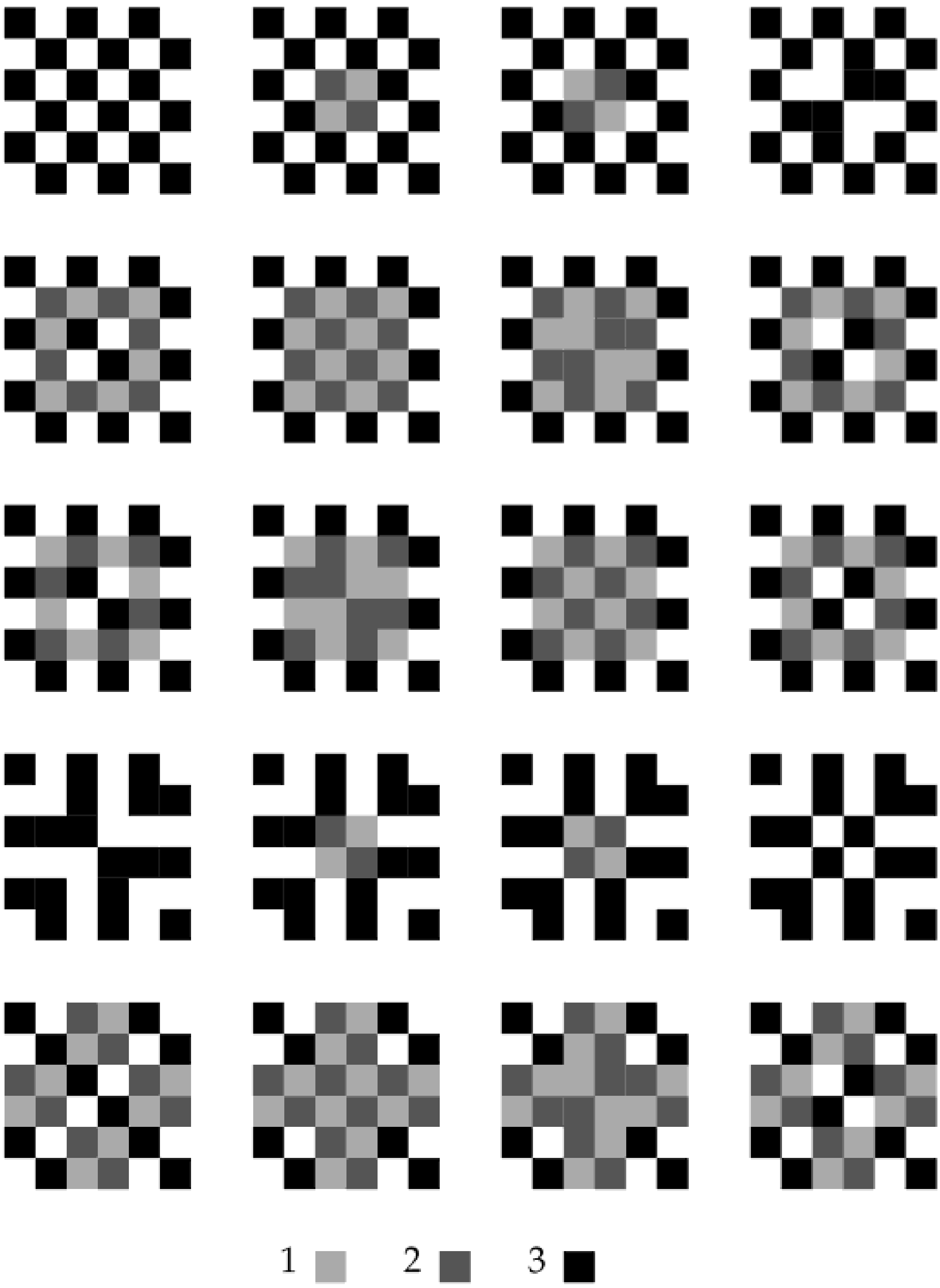


Figure 2.30 (first part)

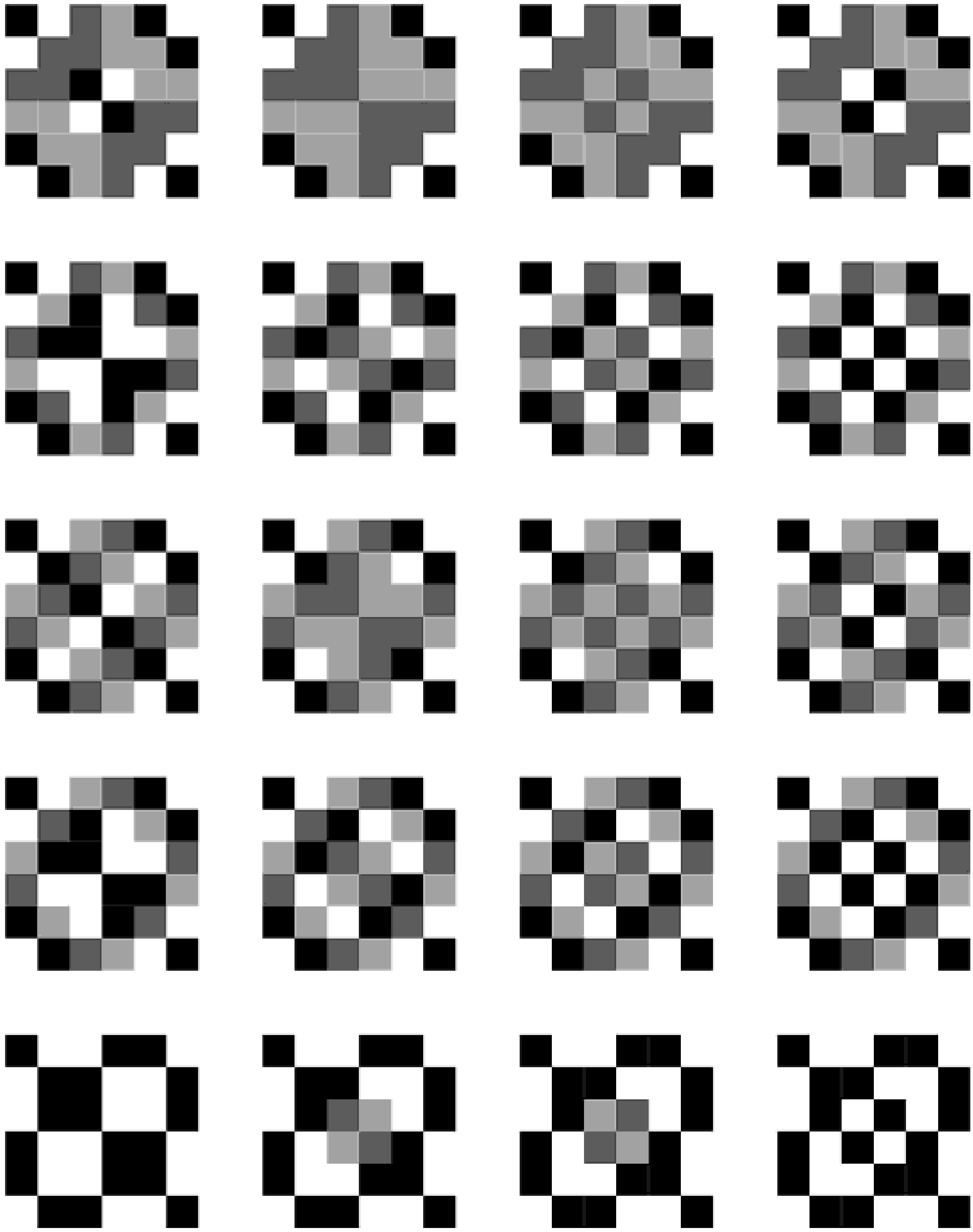


Figure 2.30 (second part)

Figure 2.30 displays the 3x3 Lunda-3-designs of the type *d4'*, with white (= 0) being the color of the first unit square [with vertices (0,0), (1,0), (1,1), and (0,1)]. Figure 2.31 shows an example of an 8x8 Lunda-4-design. Figure 2.32 displays examples of a 4x4 Lunda-5-design, an 8x4 Lunda-5-design, and a 5x3 Lunda-4-design. This time the color chosen for the first unit square is different from white.

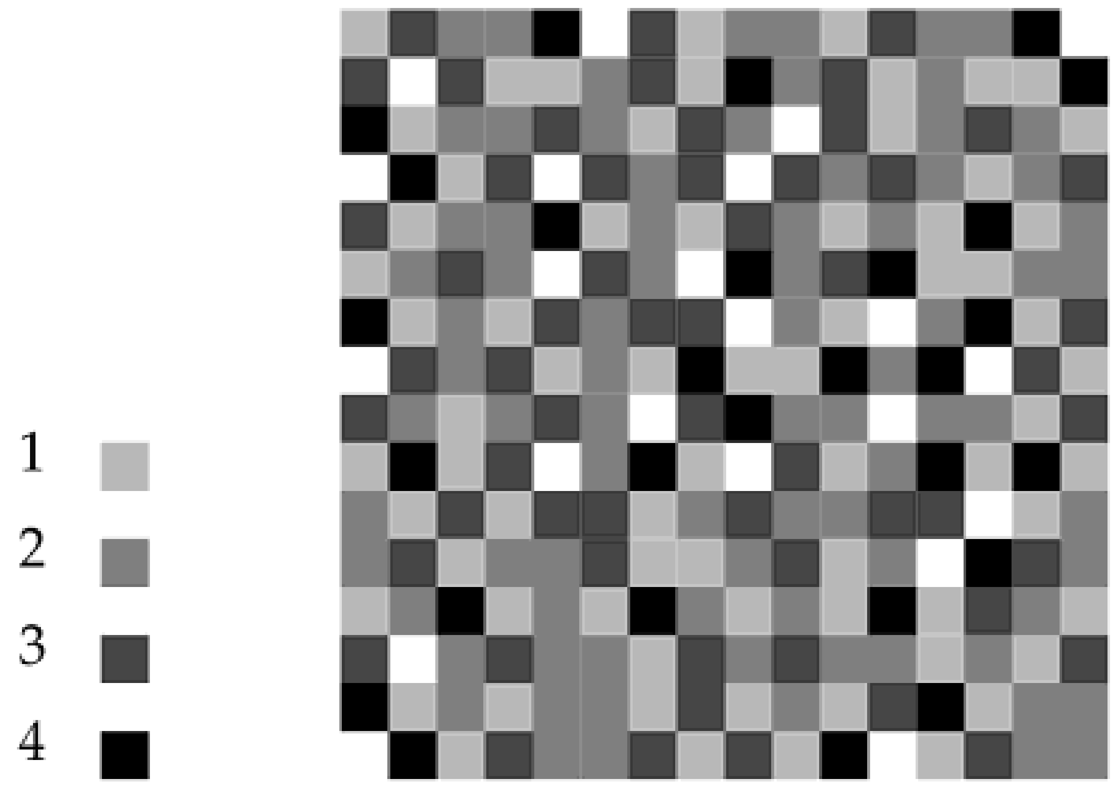


Figure 2.31

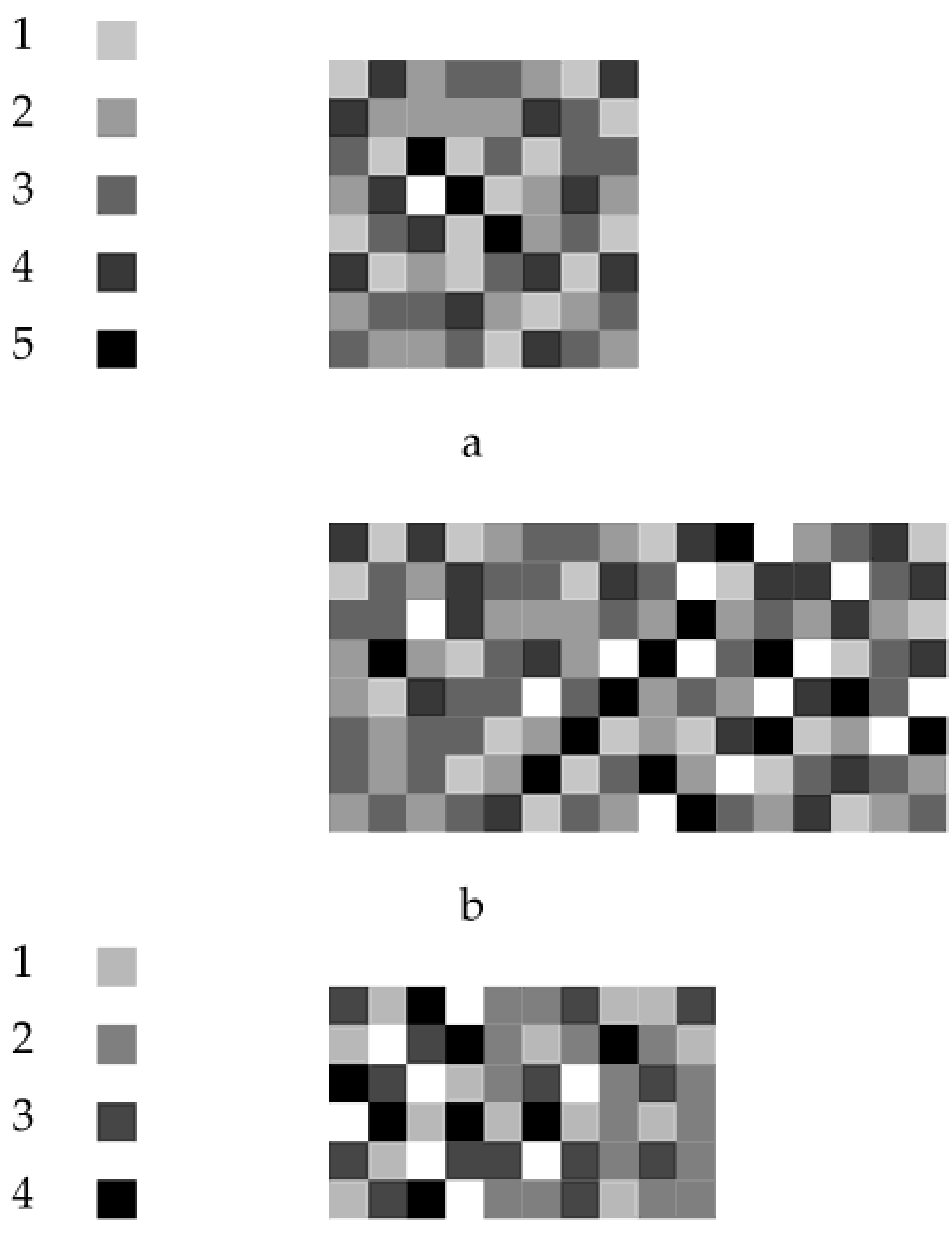


Figure 2.32

2.8 Hexagonal Lunda-designs

Another way to expand the concept of Lunda-design is to start with hexagonal grids instead of rectangular ones. Figure 2.33 displays two examples.

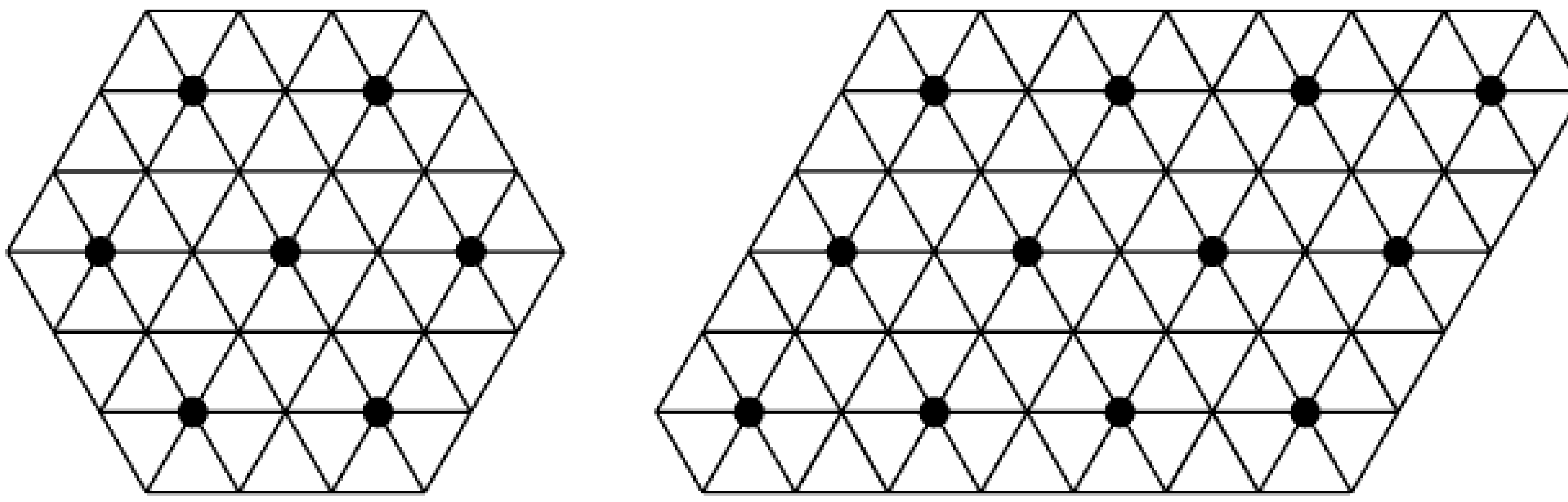


Figure 2.33

Each grid point is surrounded by six unit triangles. Each border grid point has three unit triangles that touch the border (see Figure 2.34a), and between two arbitrary neighboring grid points, there is always a hexagon composed of six unit triangles (see Figure 2.34b).

a b

Figure 2.34

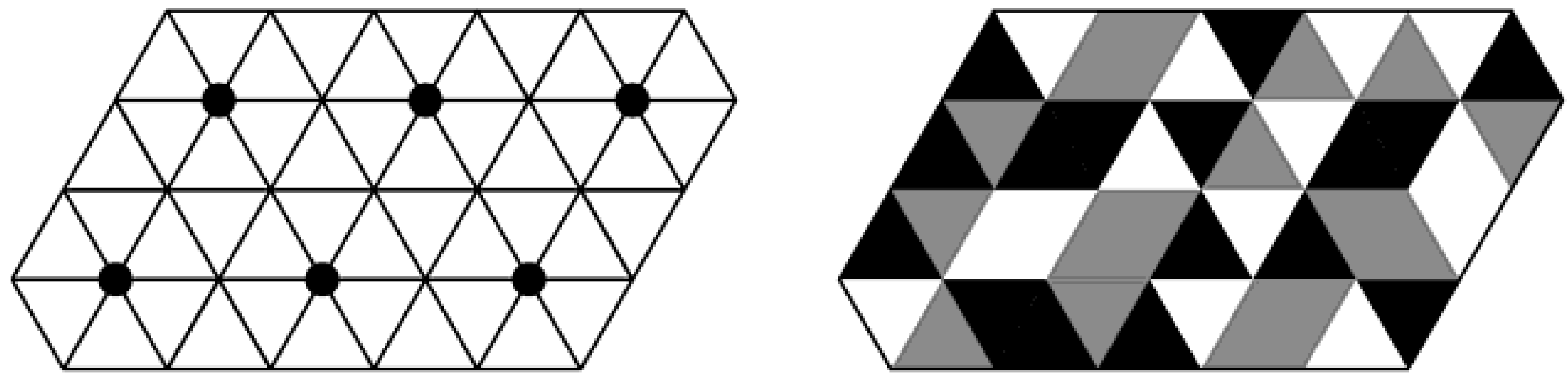
Suppose that to each unit triangle of a hexagonal grid we assign one of three colors (e.g. white, grey, and black). Then we obtain a three-colored design. If such a design satisfies the following two conditions:

(i) To the three border unit triangles of any border grid point different colors are assigned;

(ii) Of the six unit triangles between two arbitrary neighboring grid points, there are two of each color,

we call it a **hexagonal Lunda-design**.

Properties (i) and (ii) guarantee local equilibrium between the three colors. Figure 2.35b shows an example of a hexagonal Lunda-design.



a b

Figure 2.35

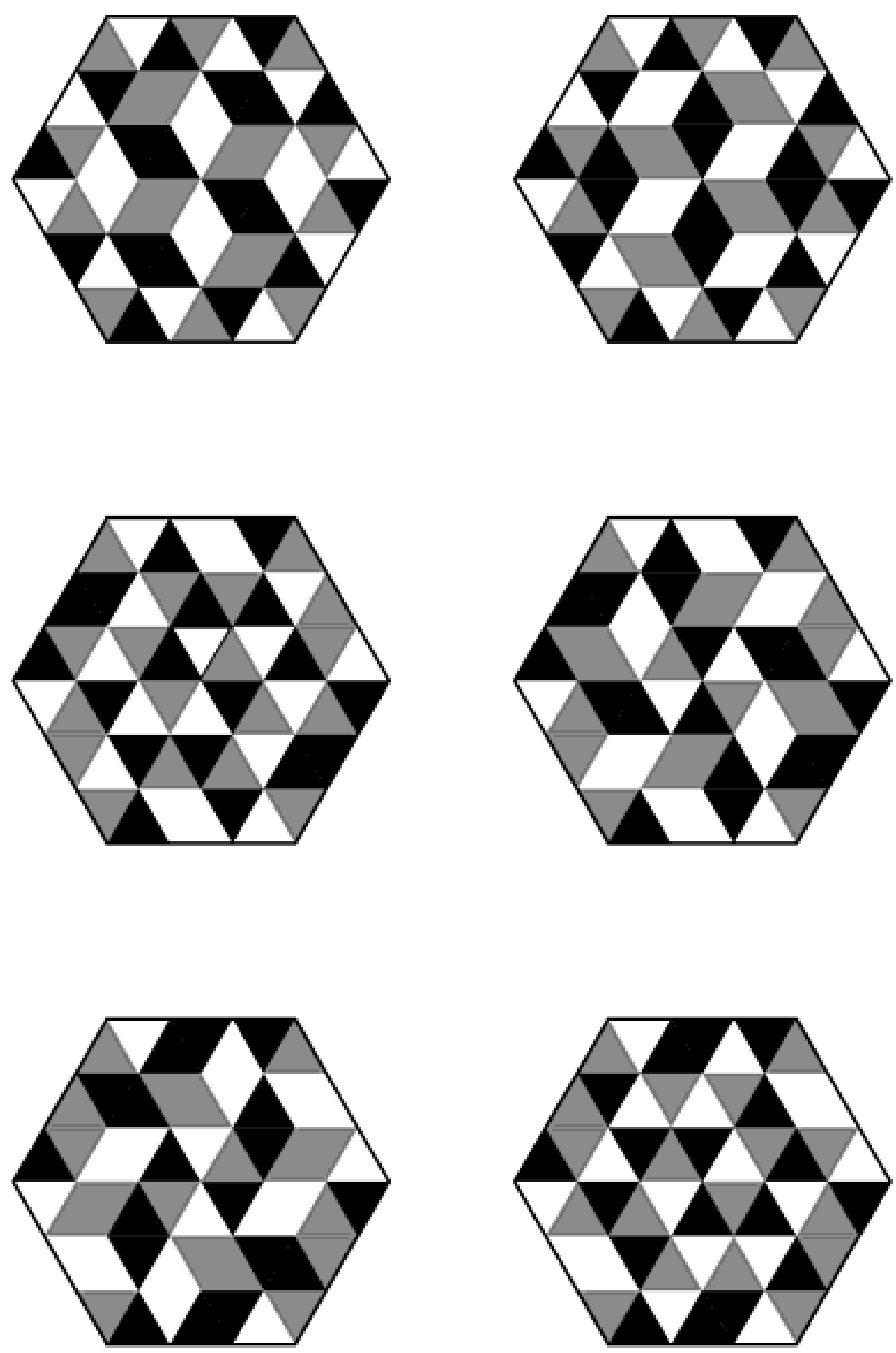


Figure 2.36

Figure 2.36 presents seven hexagonal Lunda-designs that have a three-color rotational symmetry: a 60^{o} rotation about the centre is consistent with color. A clockwise rotation by 60^{o} moves all the white to coincide with all the grey, moves grey to black, and black to white. In other words, the three colors occupy equivalent parts of the design.

In the case of the four hexagonal Lunda-designs in Figure 2.37, a clockwise rotation by 120° moves white to grey, grey to black, and black to white.

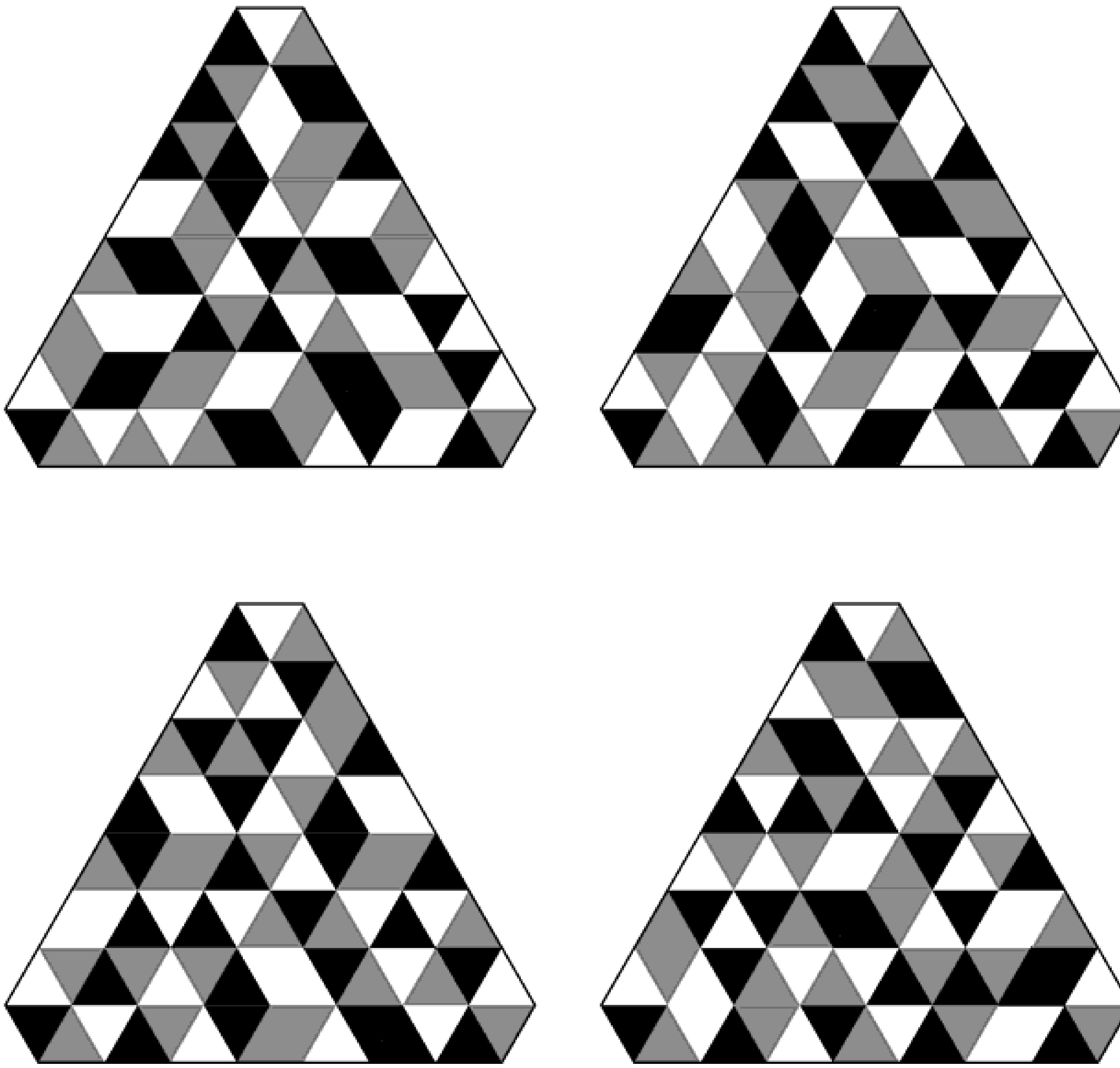


Figure 2.37

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Gerdes, Paulus (1989), Reconstruction and extension of lost symmetries: examples from the Tamil of South India, *Computers and Mathematics with Applications*, Vol. 17, N° 4-6, 791-813

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___ (1995), Extensions of a reconstructed Tamil ring-pattern, in: Pickover, C. (Ed.), *The Pattern Book: Fractals, Art, and Nature*, Singapore: World Scientific, 377-379

Jablan, Slavik (1995), Mirror generated curves, *Symmetry: Culture and Science*, Vol.6, No.2, 275-278

Lauwerier, Hans (1991), *Fractals, images of chaos*, London, Penguin Books, 209 pp.

Washburn, Dorothy & Crowe, Donald (1988), *Symmetries of Culture. Theory and Practice of Plane Pattern Analysis*, Seattle: University of Washington Press, 299 pp.

Chapter 3

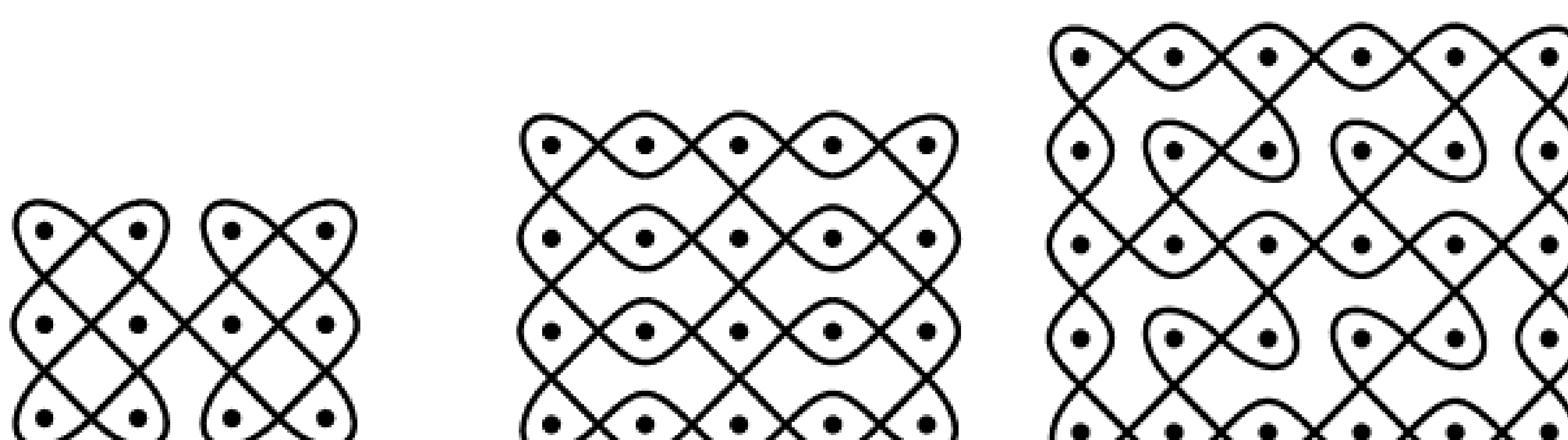
ON MIRROR CURVES AND LUNDA-DESIGNS [1]

Abstract

Some aspects of a class of curves that may be considered as generated by mirror designs are presented, with examples from several cultures. These mirror curves generate in turn, interesting black-and-white designs called Lunda-designs. The paper presents examples of these, discusses some of their properties and suggests generalizations of the concept.

3.1 Mirror designs and mirror curves

Storytellers among the Cokwe and neighboring peoples in eastern Angola and northwestern Zambia were used to illustrate their fables with standardized drawings in the sand. Such a drawing consists of a line figure that embraces all the points of an orthogonal grid of equidistant points. When analyzing this tradition, I found that several sand drawings (see the examples in Figure 3.1) may be generated in the following way [Gerdes, 1993, chap. 6].



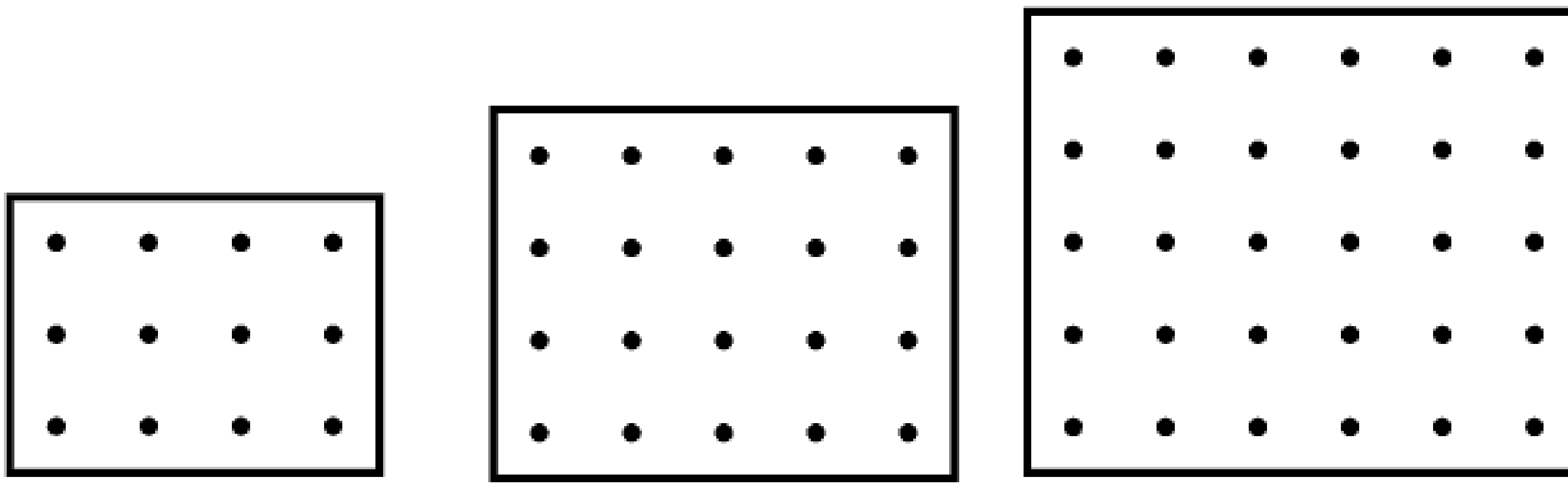
a b c

Examples of Cokwe sand drawings

Figure 3.1

[1] Published in: *Computers and Graphics, An international journal of systems & applications in computer graphics*, Oxford, 1997, Vol. 21, N° 3, 371-378.

Consider a rectangular grid RG[m,n] with vertices (0,0), (2m,0), (2m,2n), and (0,2n) and having as points (2s-1, 2t-1), where s= 1,...,*m*, and t= 1,...,*n*, and *m* and *n* are two arbitrary natural numbers (see the examples in Figure 3.2).



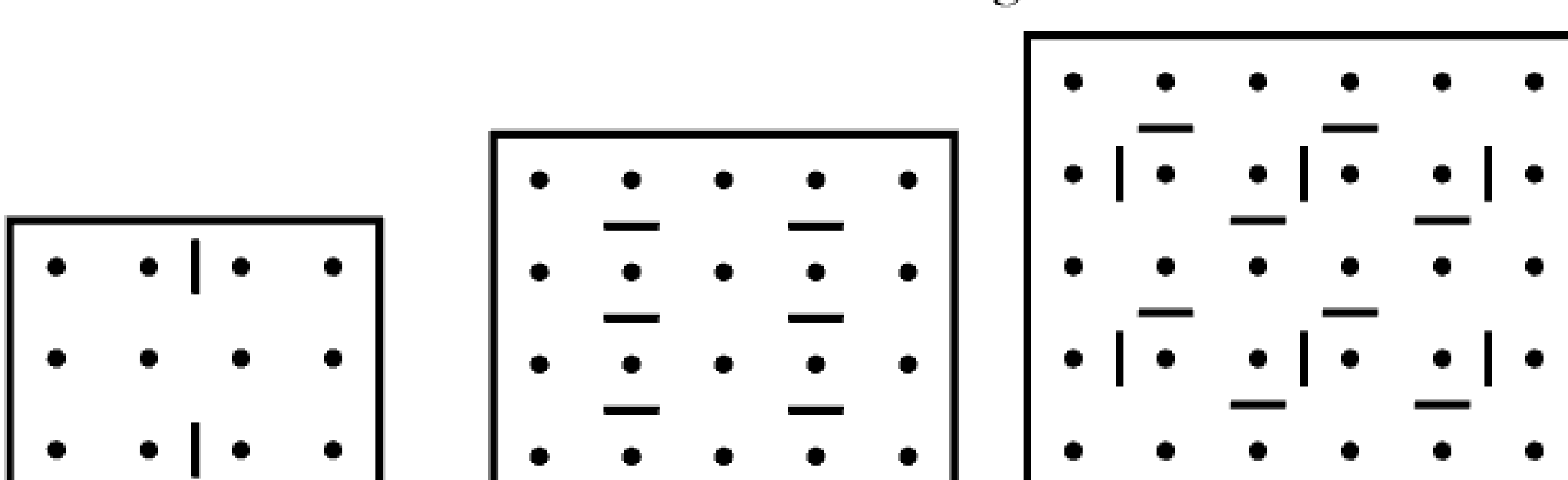
RG[4,3] RG[5,4] RG[6,5]

a b c

Rectangular grids

Figure 3.2

A curve like that shown in Figure 3.1a is the smooth version of a closed polygonal path described by a light ray emitted from the point (1,0), making an angle of 45 degrees with the sides of the rectangular grid RG[m,n]. The ray is reflected on the sides of the rectangle and on its way through the grid it may encounter double-sided mirrors, which are placed horizontally or vertically in the centre between two (horizontal or vertical) neighboring grid points. Figure 3.3 shows the position of the mirrors in order to generate the curves of Figure 3.1. These curves are rectangle-filling in the sense that they 'embrace' all the grid points. Such curves we will call (rectangle-filling) *mirror curves*. The rectangular grids together with the mirrors, which generate the curves will be called *mirror designs*.

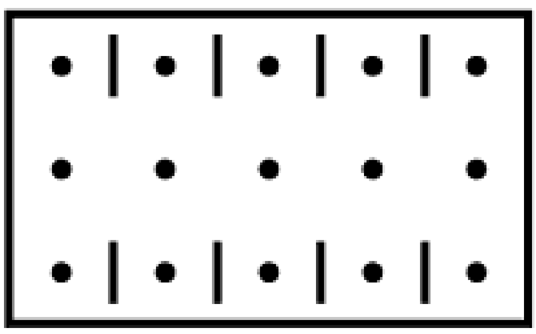


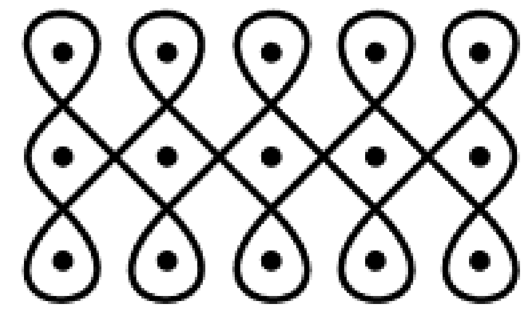
a b c

Mirror designs

Figure 3.3

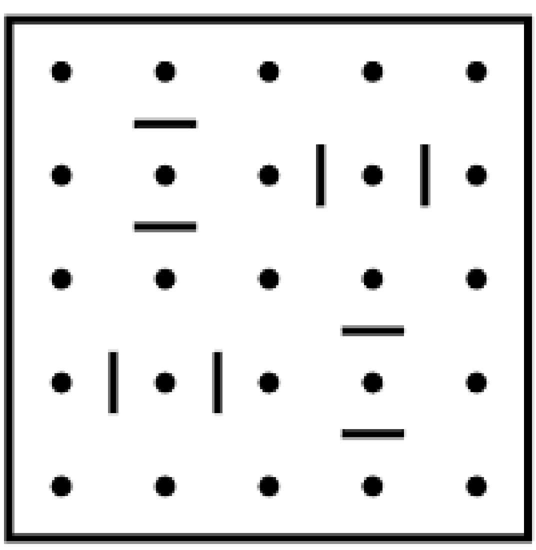
Mirror curves may also be found in other cultures. For instance, Figure 3.4a1, a2 and a3 display the mirror designs that lead to an ancient Egyptian scarab design (Figure 3.4b1), to the threshold design from the Tamil (Southern India) in Figure 3.4b2 and to the mosque decoration in Figure 3.4b3 (cf. [Gerdes, 1989], [Gerdes, 1993, chap. 9, 11]; [Gerdes, 1995]; [Hessemer, pl.46]).

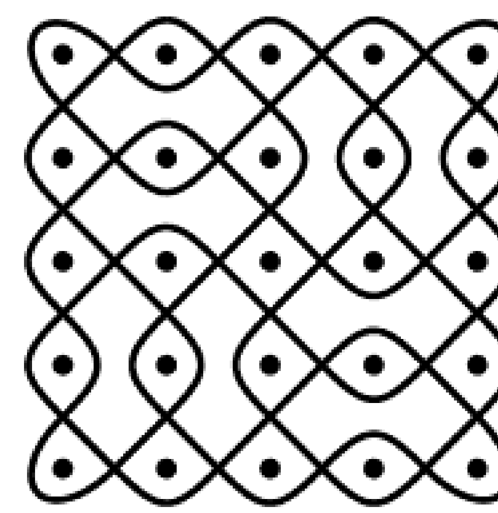




Ancient Egyptian design

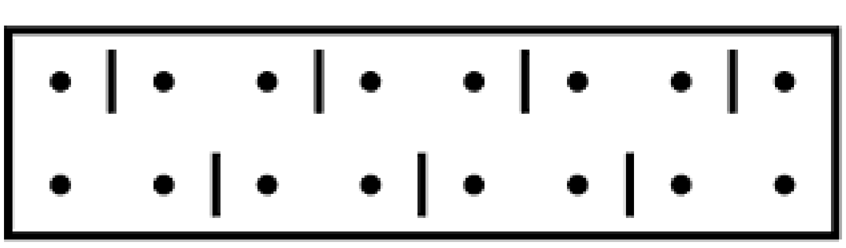
a1 b1

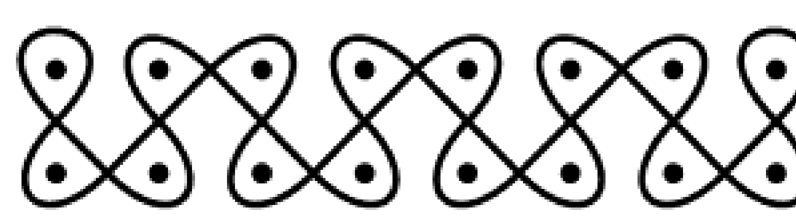




Tamil design

a2 b2





Mosque ornamentation in Cairo

a3 b3

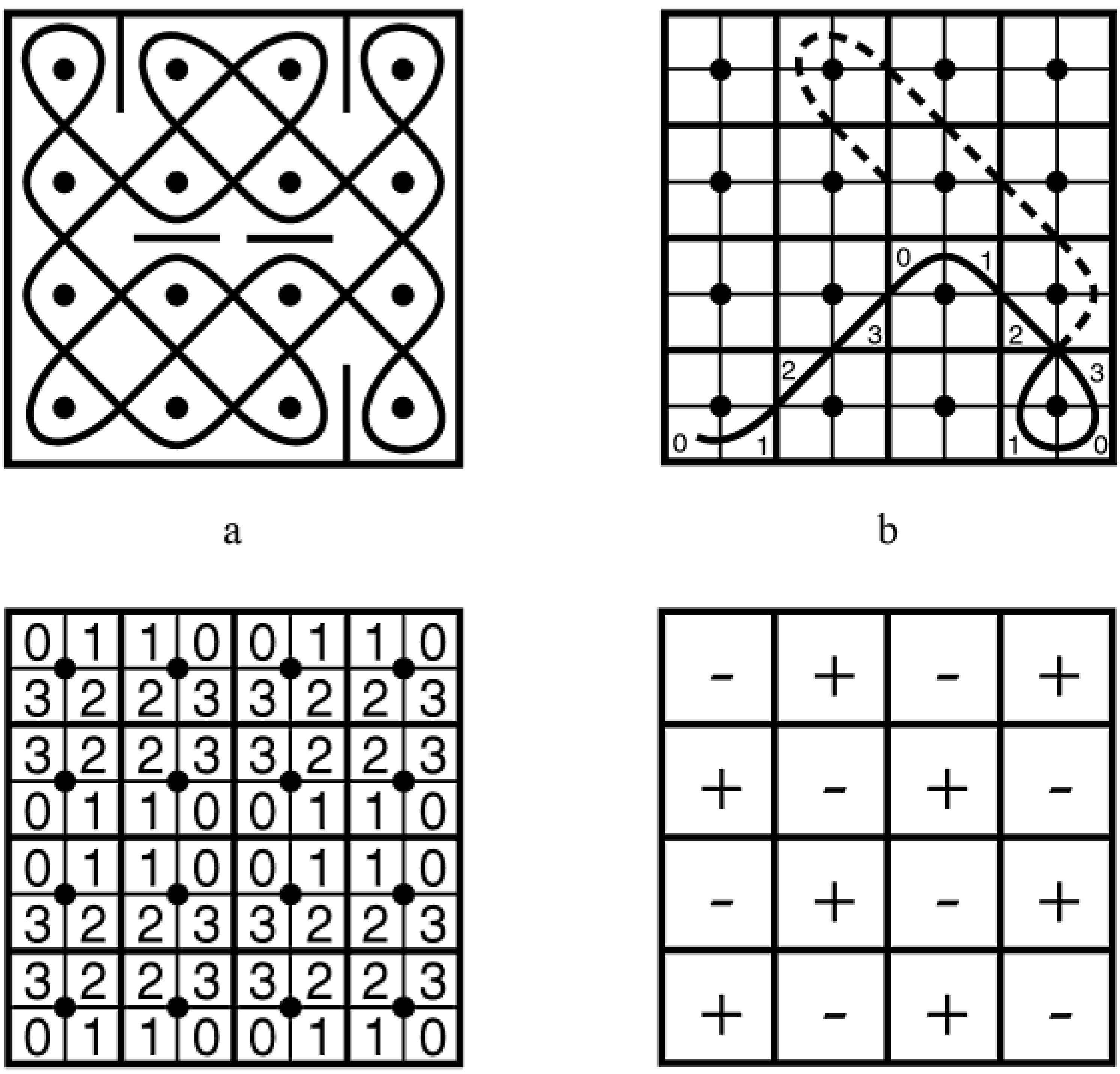
Examples of mirror curves in various cultures

Figure 3.4

Gerdes (1990; 1993, chap. 4-8) and Jablan (1995, 1996) analyze several properties and classes of mirror curves. Jablan also establishes links with the theory of cellular automata, Polya's enumeration theory, combinatorial geometry, topology, and knot theory.

3.2 Regular mirror curves

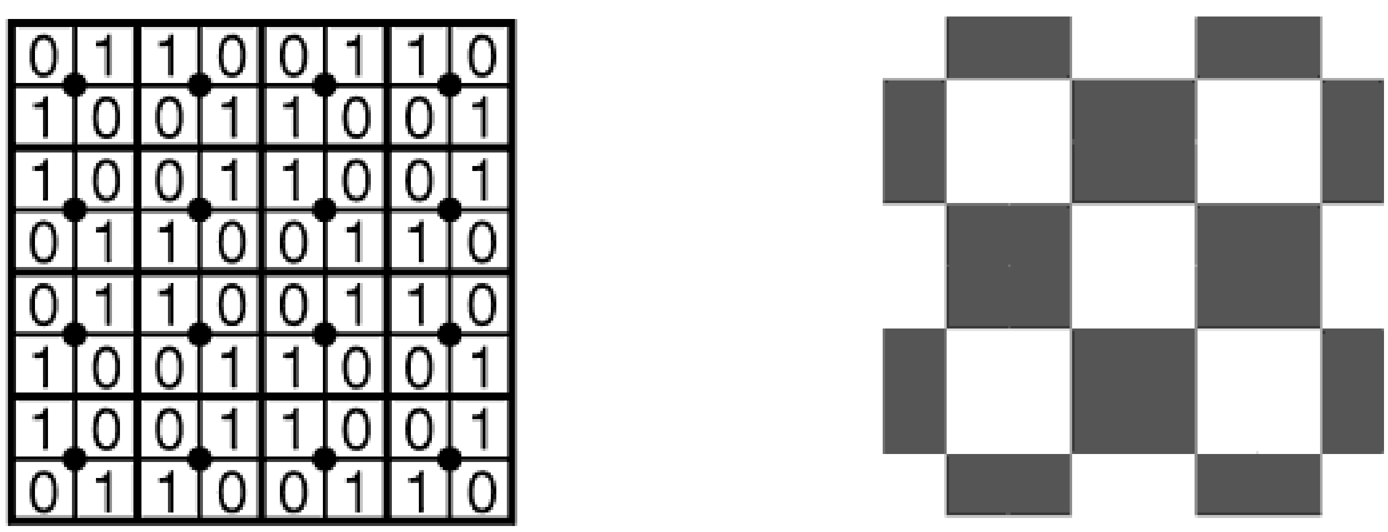
The mirror curves in Figures 3.1 and 3.4 are *regular* in the sense that in the corresponding mirror designs the horizontal mirrors are always in the centre between vertically neighboring grid points, and the vertical mirrors are always in the centre between horizontal neighboring grid points. Regular mirror curves possess some interesting properties, such as the following.



Example of enumerating the unit squares modulo 4

Figure 3.5

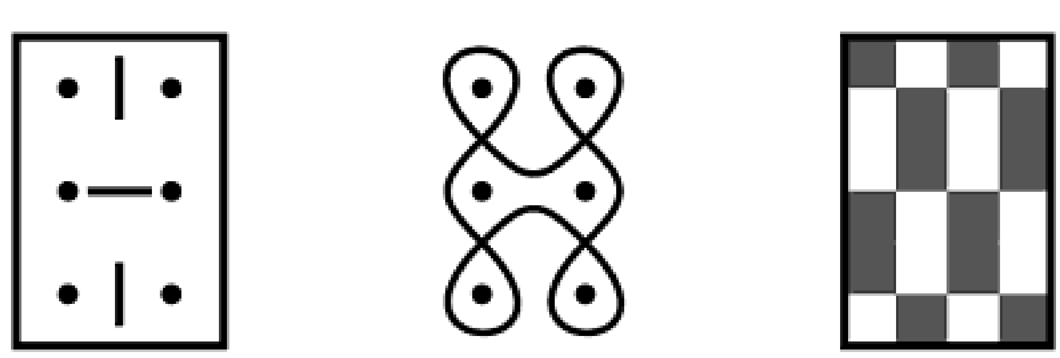
Consider a rectangle-filling, regular mirror curve. It passes precisely once through each of the unit squares of the rectangular grid. This enables us to enumerate the unit squares through which the curve successively passes 1, 2, 3, 4,...,4mn. Enumerating them modulo 4, i.e. 1, 2, 3, 0, 1, 2, 3, 0, ... a (1, 2, 3, 0)-matrix is obtained. When we start the enumeration with the unit square with vertices (0,0), (1,0), (1,1), and (0,1), we find that the four unit squares around any grid point are always numbered clockwise (negative rotation) or counter-clockwise (positive rotation) 1, 2, 3, 0. Moreover, positive and negative rotations alternate like the checkers of a chessboard (see the example in Figure 3.5). When we count the unit squares modulo 2, i.e. 1, 0, 1, 0, 1, 0, etc. , a (1,0)-matrix is obtained, or, similarly, by coloring the successive unit squares alternately black (= 1) and white (= 0), a black-and-white design is produced. Figure 3.6 presents the (1,0)-matrix and the black-and-white design, which correspond to the mirror curve in Figure 3.5a (For proofs of these theorems, see Gerdes, 1993, chap. 6; cf. chap. 1).



(0,1)-matrix
a

Black-and-white design
(with the border rectangle and the grid points unmarked)
b

Corresponding (0,1)-matrix and black-and-white-design
Figure 3.6



a b c

Ancient Egyptian scarab ornamentation (b) with corresponding mirror design (a) and black-and-white design (c)
Figure 3.7

3.3 Non-regular mirror curves

The mirror curves in Figure 3.7b and 3.8b which represent an ancient Egyptian scarab ornamentation (grid points added, cf. [Gerdes, 1993, chap. 9] and [Petrie, 1934, pl.IX, nº178]) and the structure of two Celtic knot designs [Bain, p.138, 45] are not regular: in the first case there exists one horizontal mirror between two horizontal neighboring grid points (see Figure 3.7a); in the second case there exist two vertical mirrors between vertically neighboring grid points; and in the third case there are both horizontal mirrors between horizontally neighboring points, and vertical mirrors between vertically neighboring points (see Figure 3.8a). Figure 3.8c shows the black-and-white designs these mirror curves produce when we color the successive unit squares through which the curves successively pass alternately black and white (if we start the coloring at any other unit square, the final design is either the same or its negative).

a1 a2

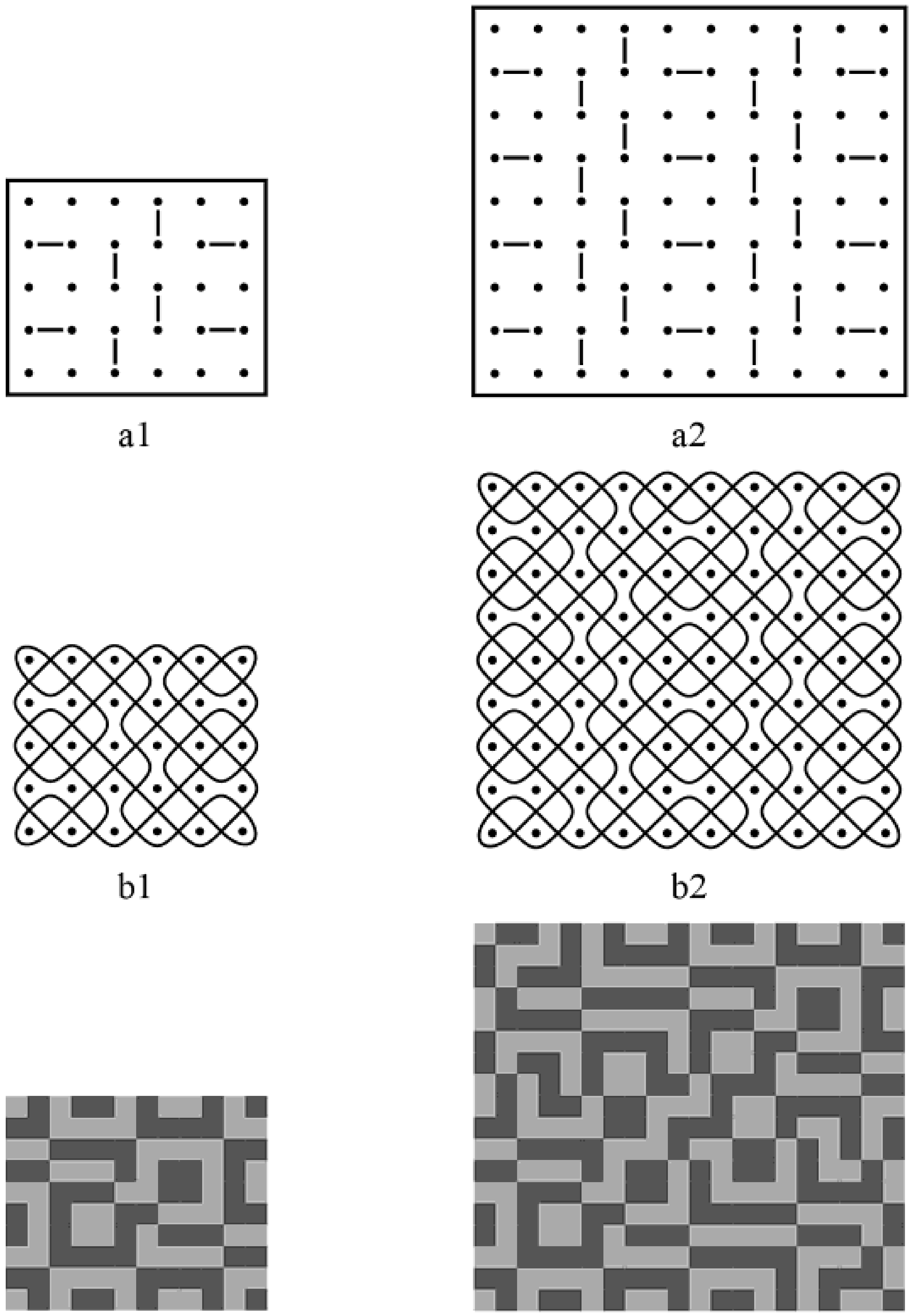
b1 b2

c1 c2

Two Celtic knot designs (b) with corresponding mirror designs (a) and black-and-white designs (c)

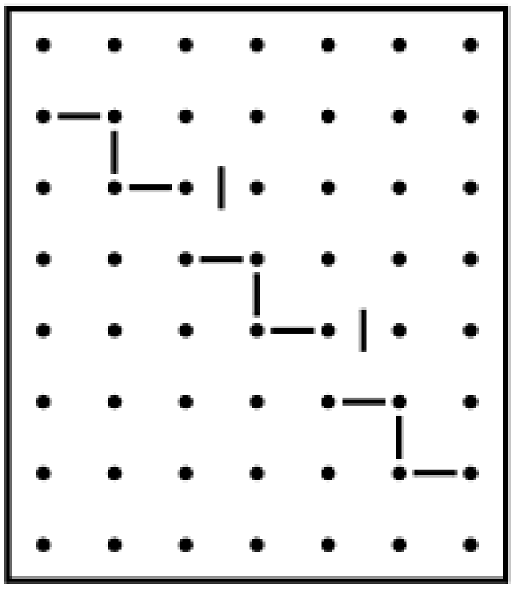
Figure 3.8

Figures 3.9 and 3.10 present further examples of non-regular mirror designs, and the mirror curves and corresponding black-and-white designs they generate. Knowing only the position of the mirrors it is difficult to conjecture how the corresponding black-and-white design will look.

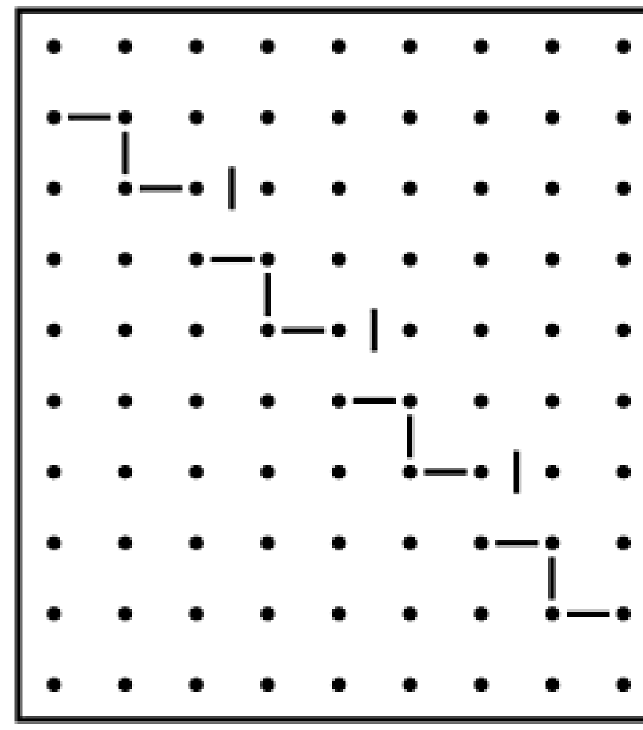


Examples of non-regular mirror designs and of the mirror curves and Lunda-designs they generate

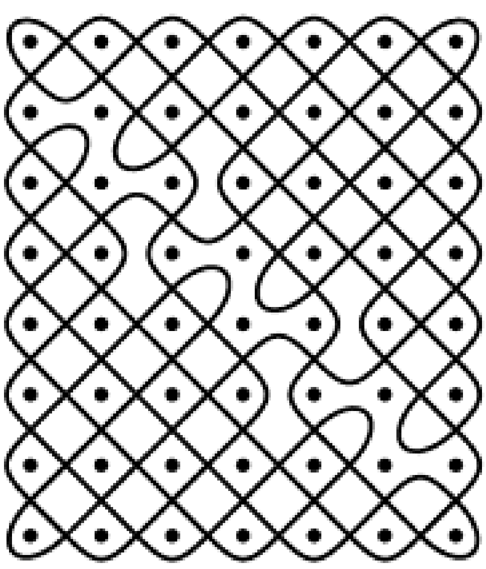
Figure 3.9



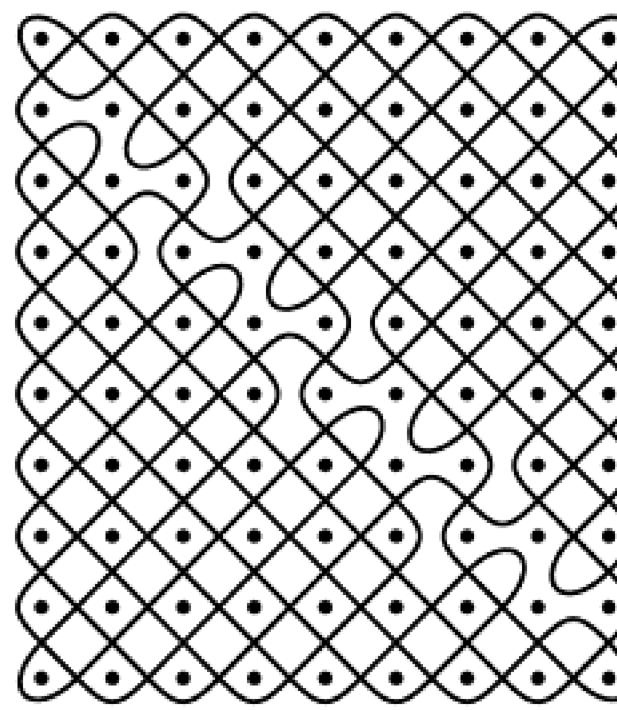
a1



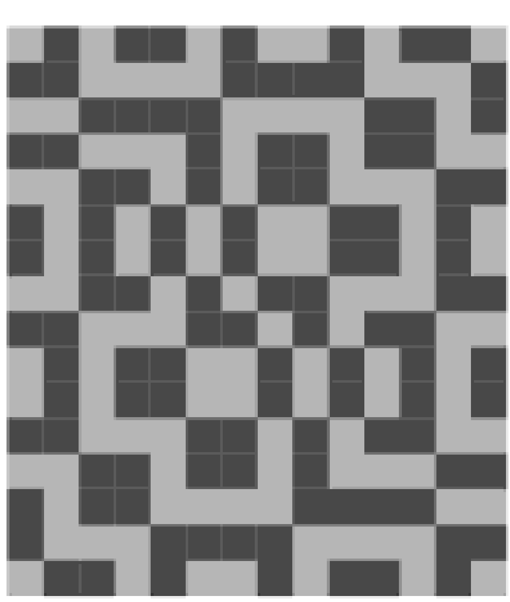
a2



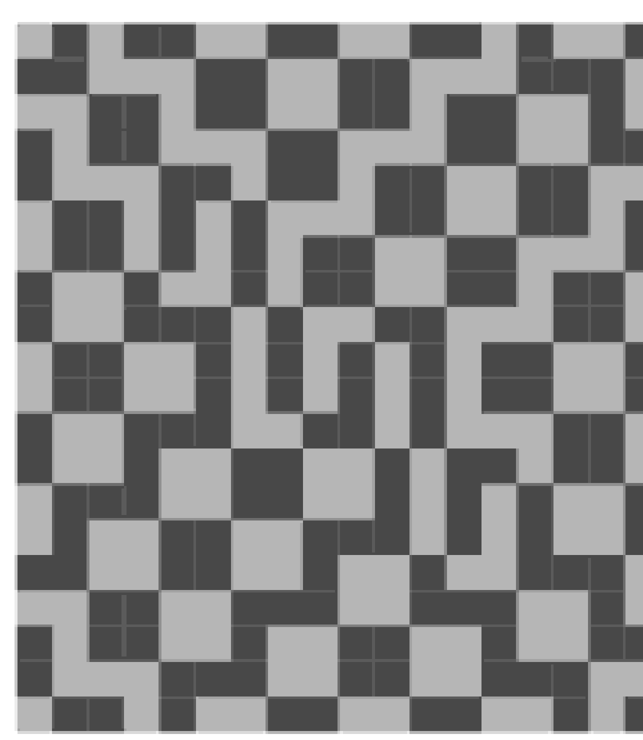
b1



b2



c1



c2

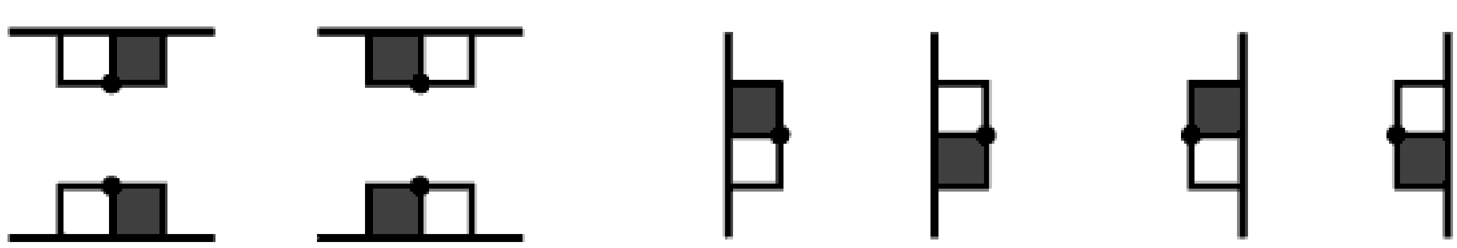
Further examples of non-regular mirror designs and of the mirror curves and Lunda-designs they generate
Figure 3.10

As this type of black-and-white design was discovered in the context of analyzing sand drawings from the Cokwe, who predominantly inhabit the northeastern part of Angola, a region called Lunda, we have given them the name of *Lunda-designs*. (Many) Lunda-designs seem to me — and to colleagues and students to whom I have shown them — aesthetically appealing. Where do possible reasons for this lie? What characteristics do Lunda-designs have in common?

3.4 Properties of Lunda-designs

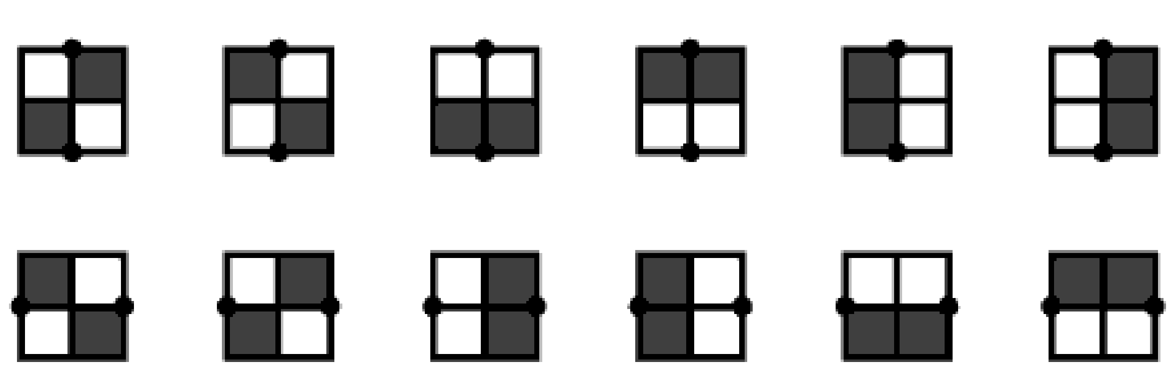
Searching for the common characteristics of Lunda-designs (of dimensions mxn), the following symmetry properties were observed and proven [see Appendix 1]:

(i) In each row there are as many black as white unit squares;
(ii) In each column there are as many black as white unit squares;
(iii) Of the two border unit squares of any grid point in the first or last row, or in the first or last column, one is always white and the other black (see Figure 3.11);



Possible border situations
Figure 3.11

(iv) Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are always black and two are white (see Figure 3.12).



Possible situations between vertical and horizontal neighboring grid points
Figure 3.12

Properties (i) and (ii) guarantee a global equilibrium between black and white unit squares for each row and column. Properties (iii) and (iv) guarantee more local equilibriums.

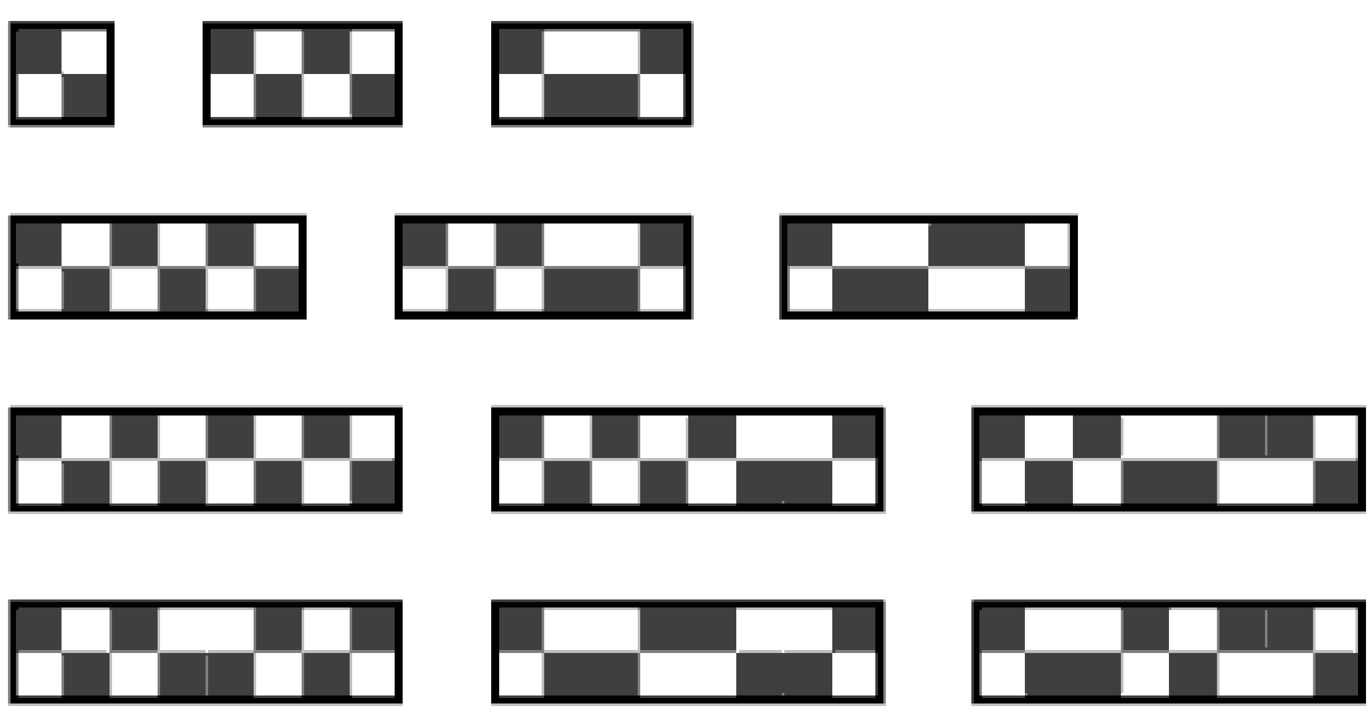
From (i) it follows that the number of black unit squares of any row is equal to m, and from (ii) that the number of black unit squares of any column is equal to *n*.

Conversely, the following theorem can be proved [see Appendix 1]:

* any rectangular black-and-white design that satisfies the properties (i), (ii), (iii), and (iv) is a Lunda-design.

In other words, for any rectangular black-and-white design that satisfies the properties (i), (ii), (iii), and (iv), there exists a (rectangle-filling) mirror curve that produces it in the discussed sense. Moreover, in each case, such a mirror curve may be constructed.

The characteristics (i), (ii), (iii), and (iv) may be used to define Lunda-designs of dimensions mxn (in brief: *mxn Lunda-designs*). In fact, it may be proven that the characteristics (iii) and (iv) are sufficient for this definition, as they imply (i) and (ii).

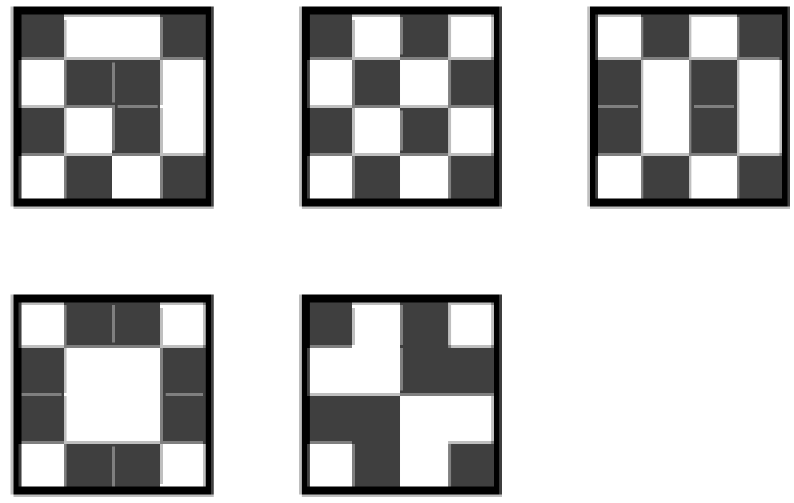


1x1, 2x1, 3x1 and 4x1 Lunda-designs
Figure 3.13

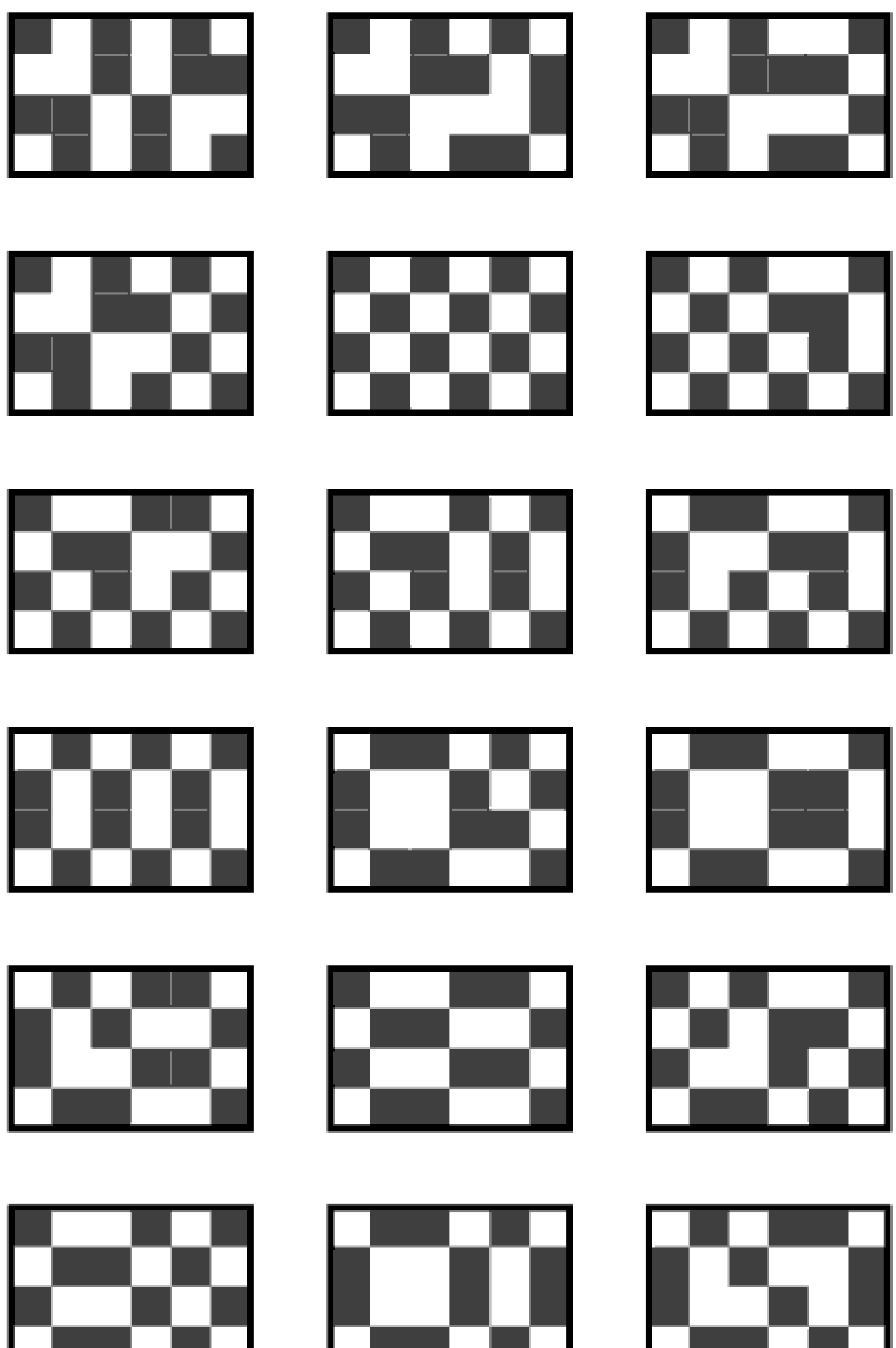
3.5 Classes of Lunda-designs

Figure 3.13 shows all distinct 1x1, 2x1, 3x1 and 4x1 Lunda-designs. We do not include designs that may be obtained from the ones presented by reflection, rotation, or by interchanging black and white. It is interesting to note that the 1x1 Lunda-design symbolizes wisdom among the Akan populations in Ghana and Côte d'Ivoire [Niangoran-Bouah, p. 210].

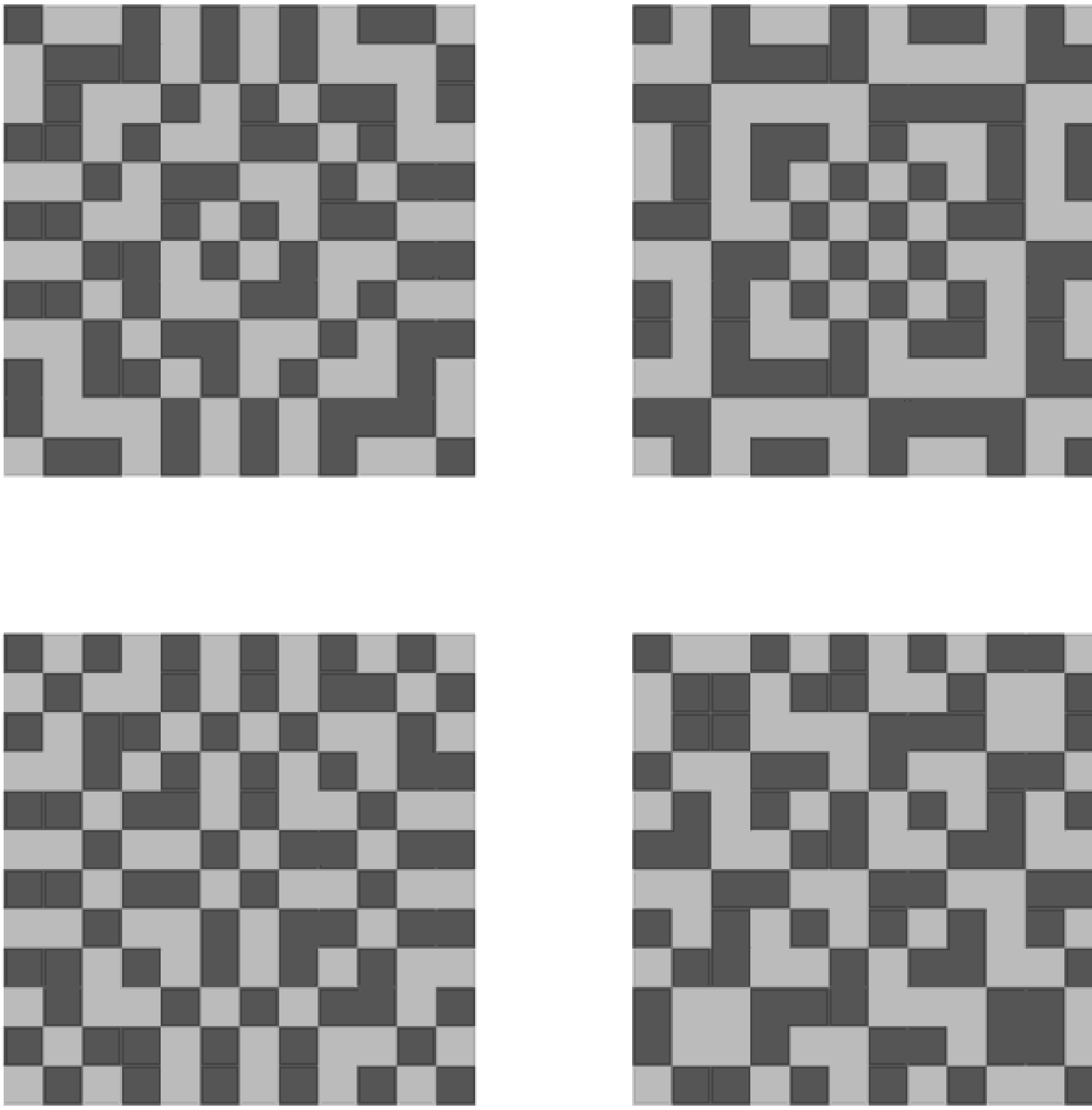
Figure 3.14 displays the 5 distinct 2x2 Lunda-designs and Figure 3.15 the 3x2 Lunda-designs.



2x2 Lunda-designs
Figure 3.14

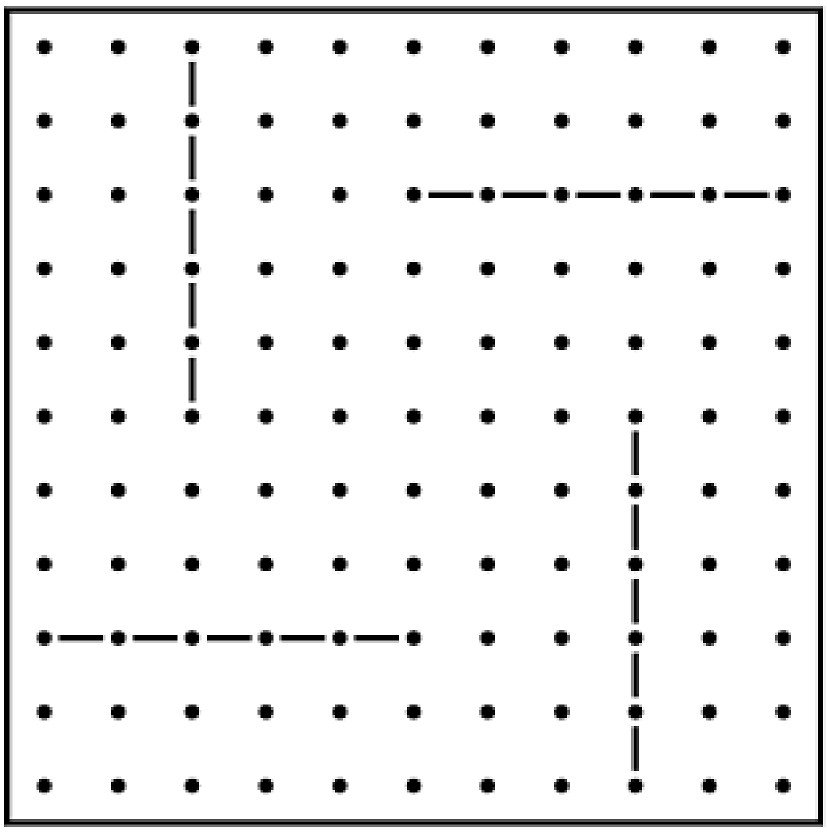


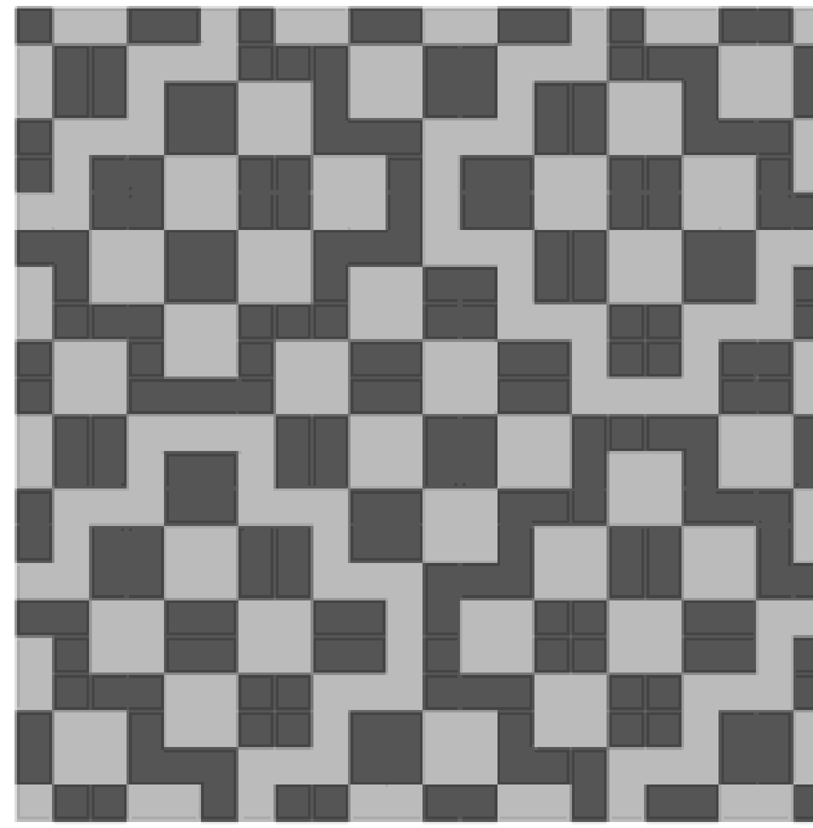
3x2 Lunda-designs
Figure 3.15



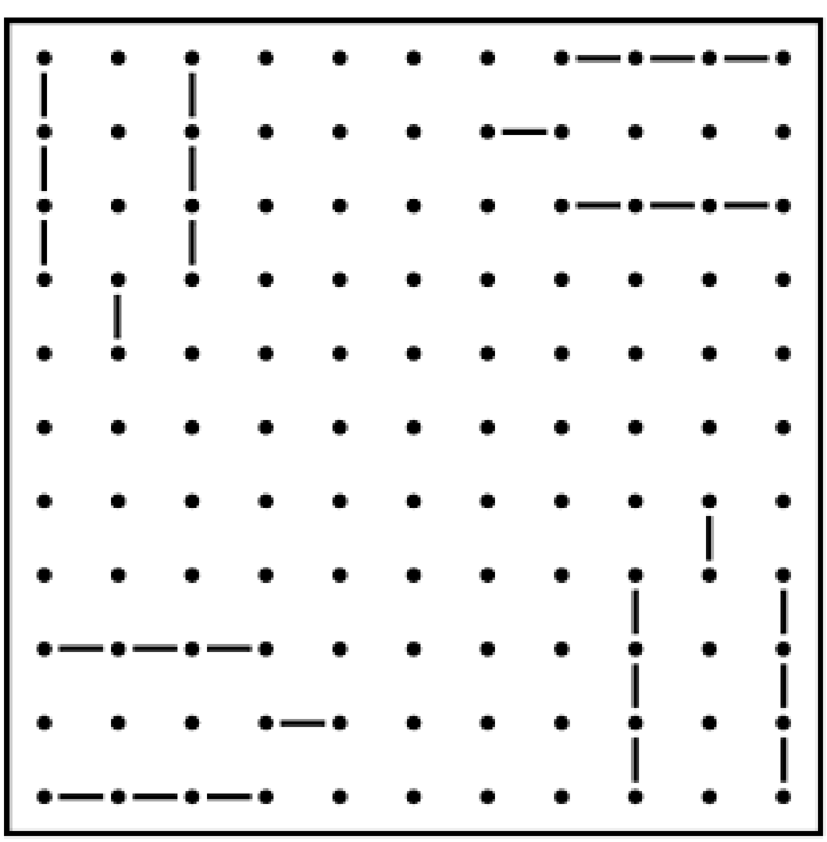
Examples of symmetrical 6x6 Lunda-designs
Figure 3.16

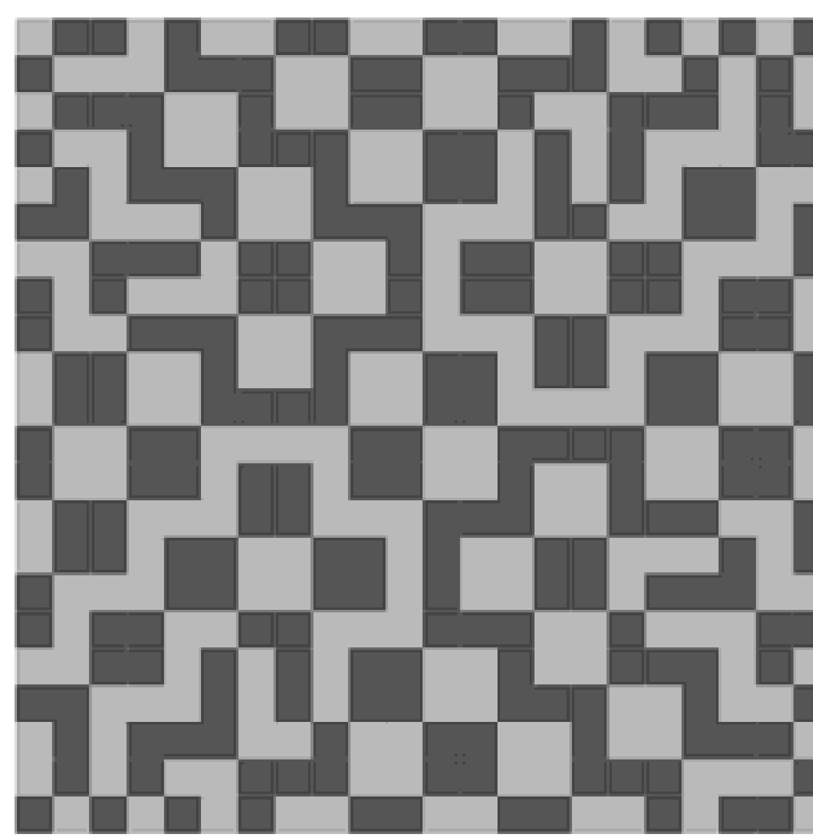
Particularly visually attractive are Lunda-designs, which display extra symmetries. Figure 3.16 presents examples of 6x6 Lunda-designs, which admit reflections in the diagonals that preserve the colors, and vertical and horizontal reflections interchanging black and white. Figure 3.17a2 presents a 11x11 Lunda-design with the same symmetries as the examples in Figure 3.16; Figure 3.17b2 displays a 11x11 Lunda-design that, although it does not have symmetry axes, possesses the property that a quarter-turn about the respective centre reverses the colors, and consequently a half-turn preserves the colors. Figures 3.17a1 and b1 show mirror designs, which generate the respective Lunda-designs.





a1 a2





b1 b2

Examples of symmetrical 11x11 Lunda-designs

Figure 3.17

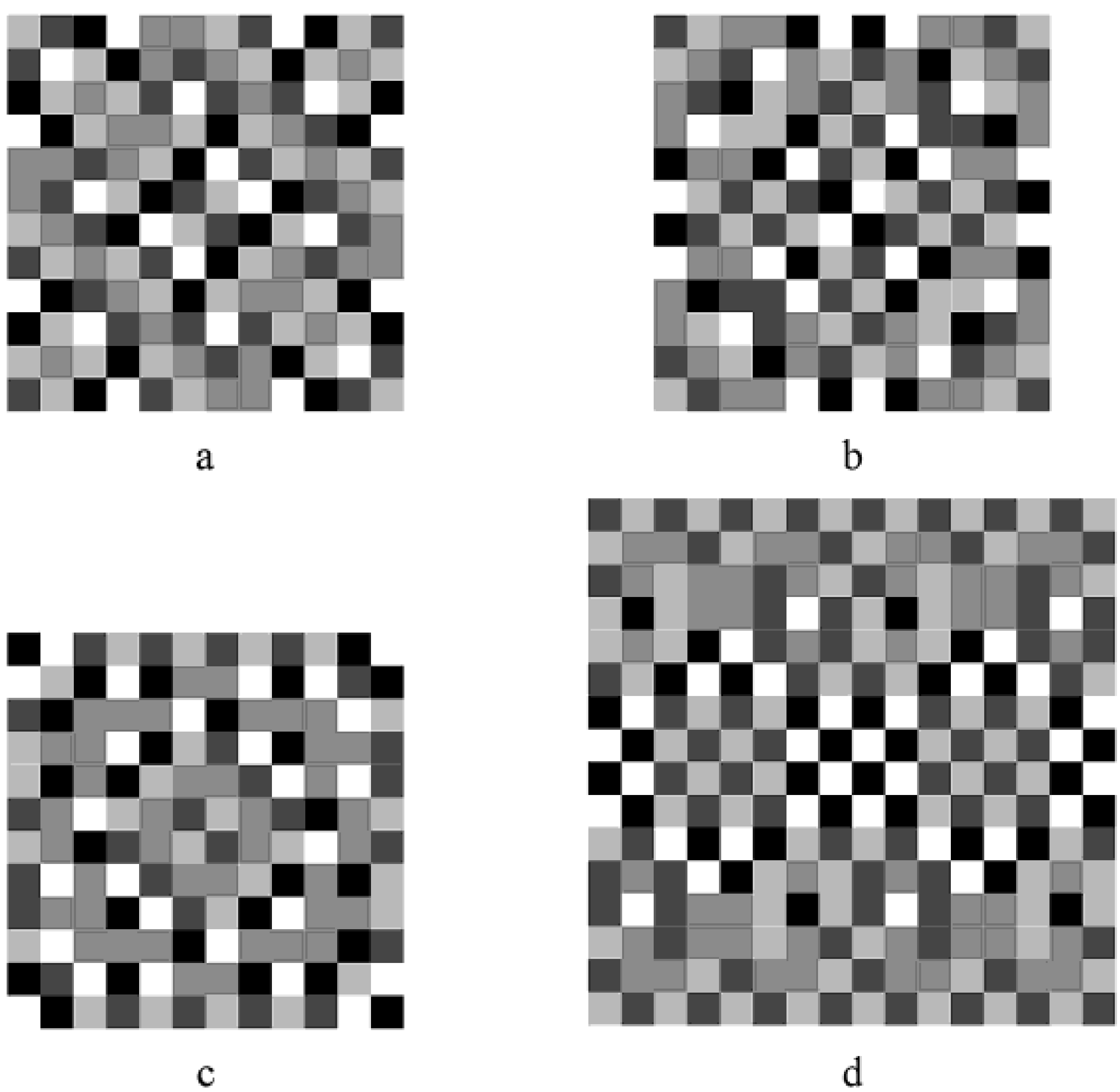
3.6 Generalizations

The concept of Lunda-design may be generalized or extended in various ways. In chapter 2 we showed, for instance, that it is possible to build up (multicolor) hexagonal Lunda-designs by starting with triangular grids. Here we will introduce Lunda-k-designs and circular and polyominal Lunda-designs.

Lunda-k-designs

As Lunda-designs may be considered as matrices, it is quite natural to define addition of Lunda-designs in terms of matrix addition. The sum of *k* mxn Lunda-designs may be called a *mxn Lunda-k-design*. The Lunda-k-designs inherit the following symmetry properties:

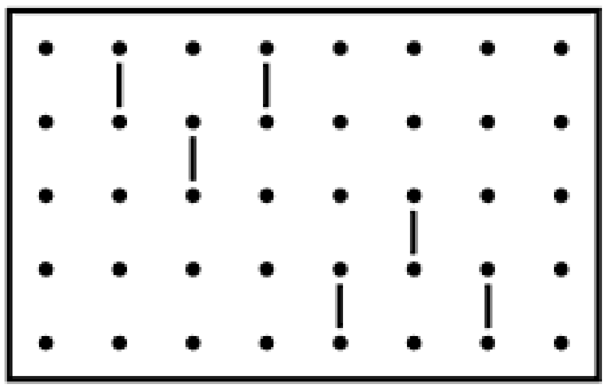
(i) The sum of the elements in any row is equal to km;
(ii) The sum of the elements in any column is equal to kn;
(iii) The sum of the integers in the two border unit squares of any grid point in the first or last rows or columns is equal to *k*;
(iv) The sum of the integers in the four unit squares between two arbitrary (vertical or horizontal) neighbor grid points is always 2*k*.



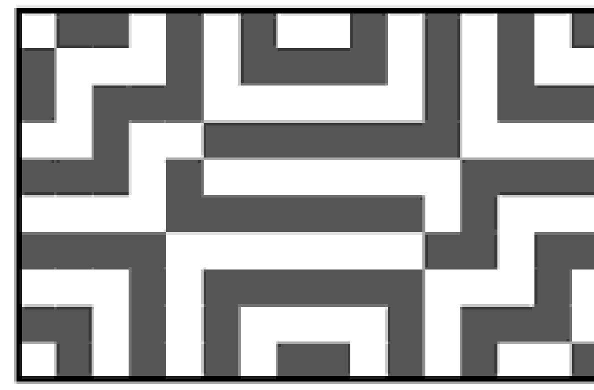
□ = 0 ▒ = 1 ▒ = 2 ▓ = 3 ■ = 4

Examples of symmetrical Lunda-4-designs
Figure 3.18

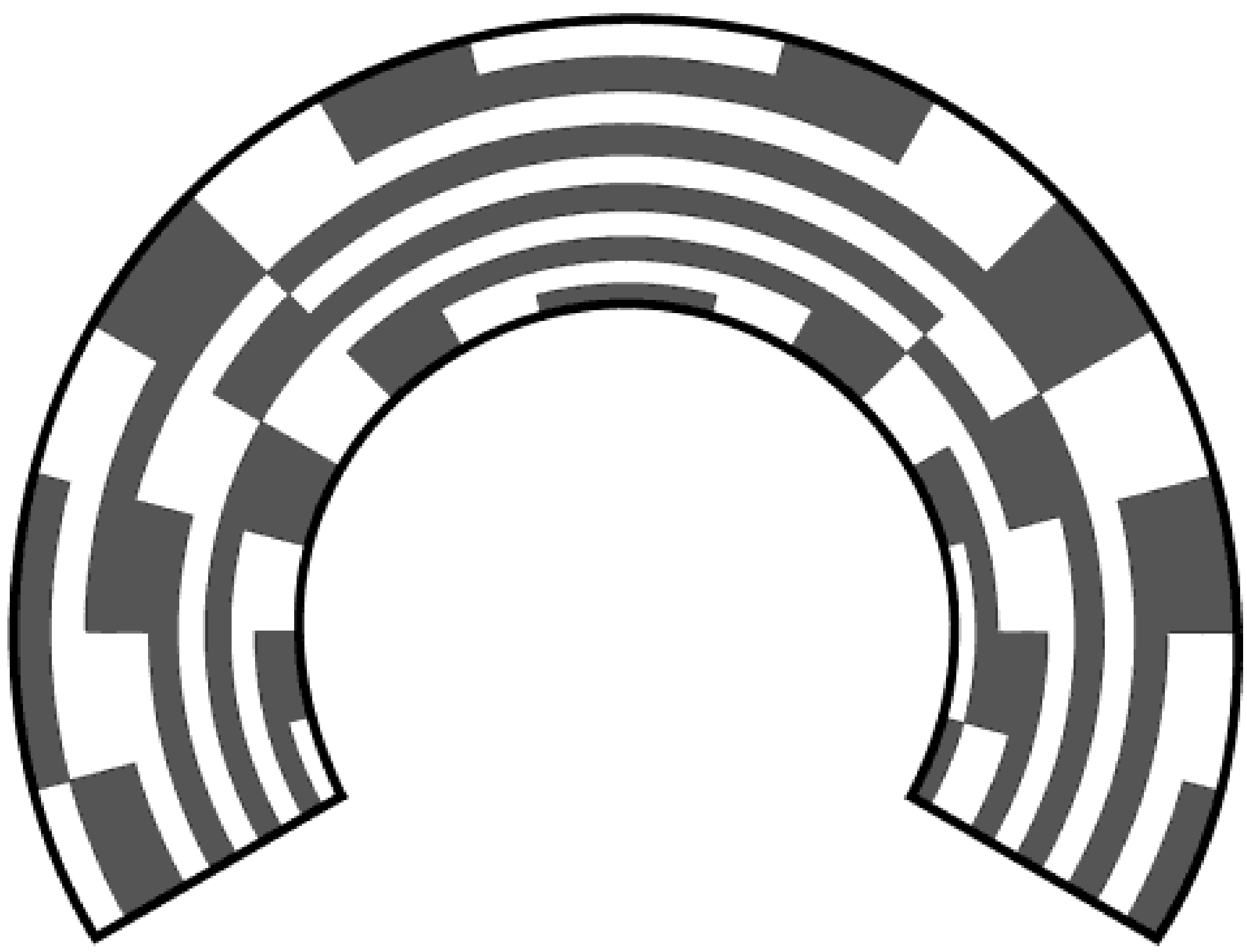
The characteristics (iii) and (iv) imply (i) and (ii), so they may be used to define mxn Lunda-k-design. Figure 3.18 displays examples of Lunda-4-designs.



a



b

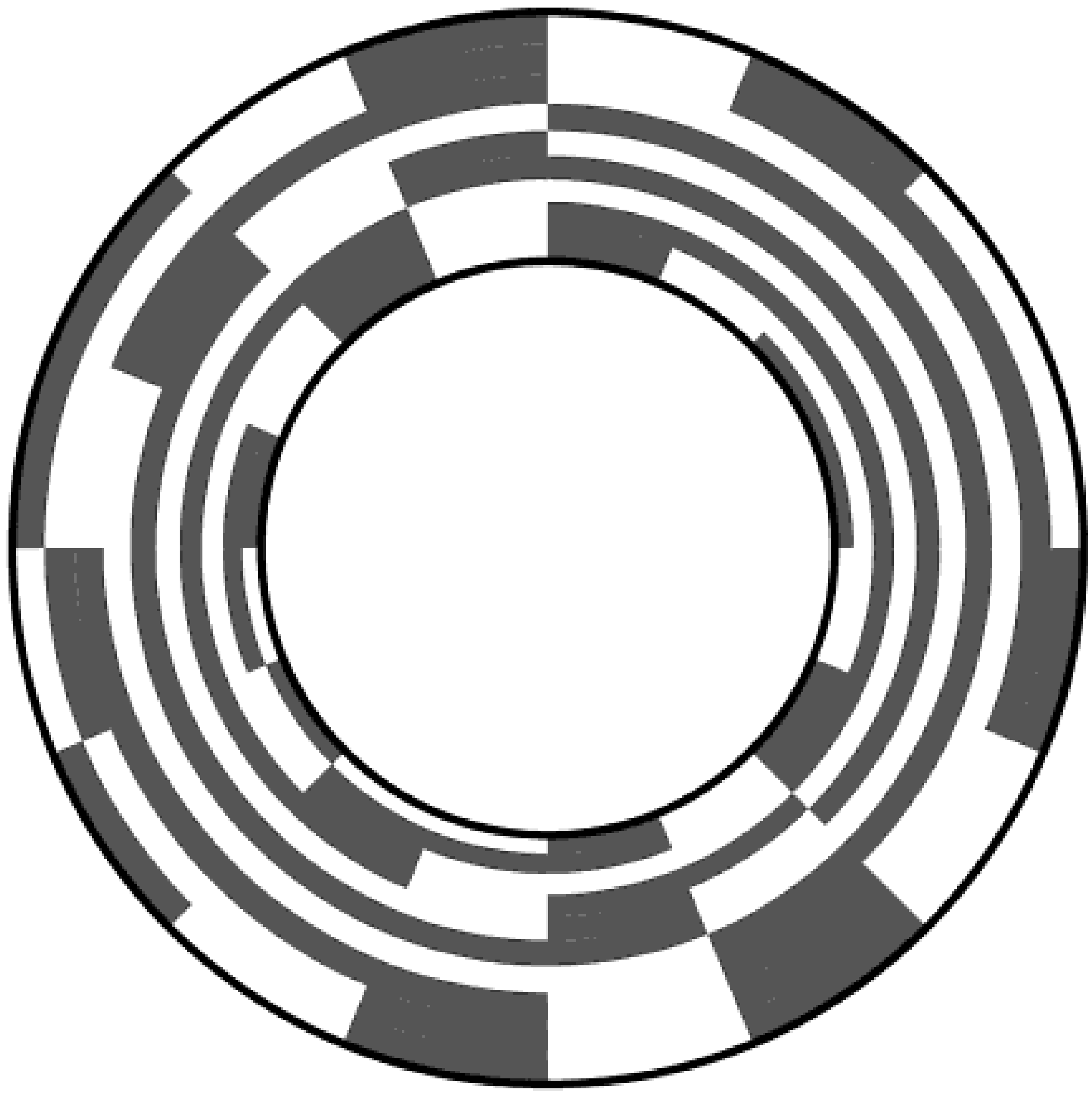


c

Example of the transformation of a rectangular Lunda-design
Figure 3.19

Circular Lunda-designs

Any Lunda-design may be topologically transformed as in the example shown in Figure 3.19. Property (iii) guarantees that, if one now joins the straight sides (see Figure 3.20), property (iv) is still valid. This leads us to the conception of circular Lunda-designs. Figure 3.21 presents examples of symmetrical 5x5 Lunda-designs together with the circular Lunda-designs they generate.

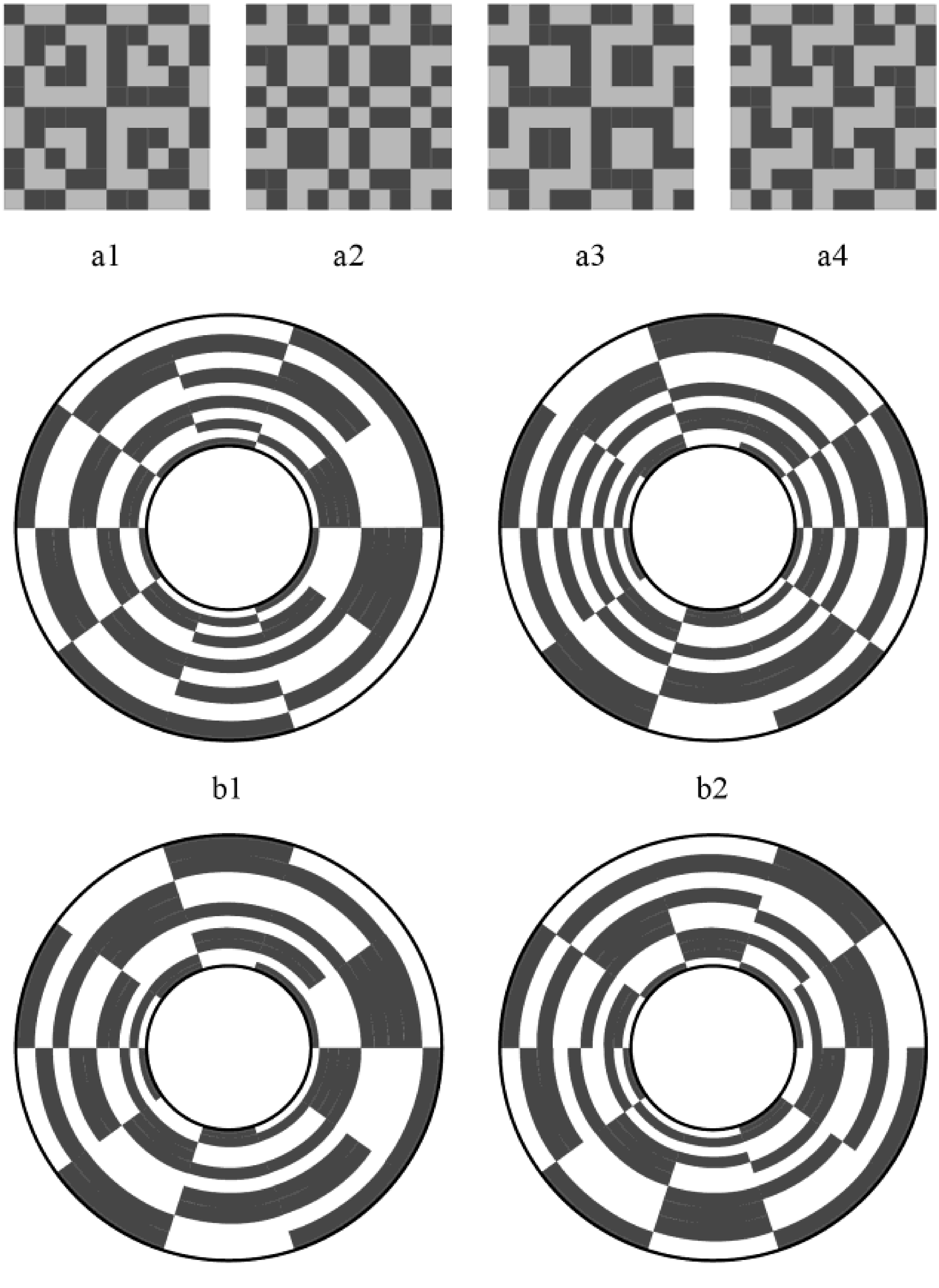


Circular Lunda-design corresponding
to the rectangular Lunda-design in Figure 3.19
Figure 3.20

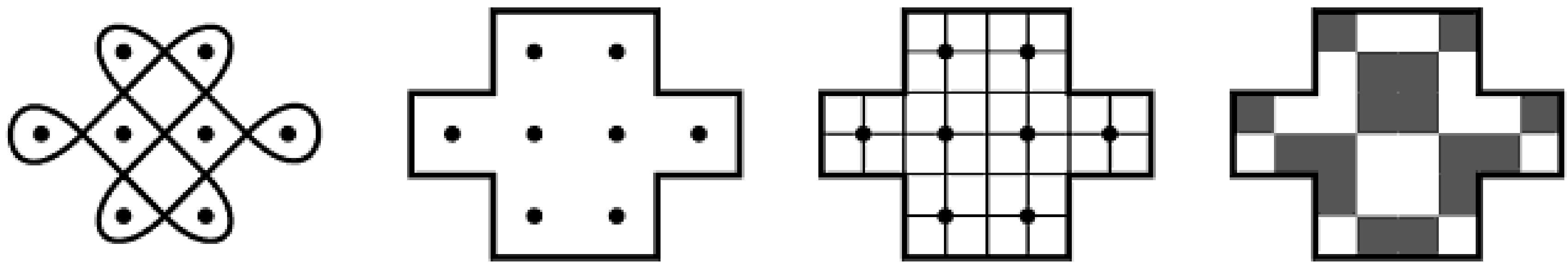
If we curve any Lunda-design in space and join two opposite sides, we obtain a black-and-white cylinder. By curving the cylinder and joining its opposite circles, we transform it into a black and white torus. This leads to the conception of cylindrical and torus Lunda-designs.

Polyominal Lunda-designs

In various cultures and historical periods there appear figures that may be considered as mirror curves, if we admit polyominal borders instead of only rectangular borders. A polyomino is a simply connected set of equal-sized squares. For instance, Figure 3.22b shows the polyomino with grid points that leads to the mirror curve (there are no internal mirrors) in Figure 3.22a. This mirror curve appears among the Cokwe in Angola, in Japan as a crest design [Adachi, p.95], in China as a lattice design [Dye, p. 99], among the Tamil in Southern India [Layard, p. 137] and also in Bhutan as a good luck symbol (W. Gibbs, personal communication, 1990). Figure 3.22d displays the Lunda-design generated by the curve.



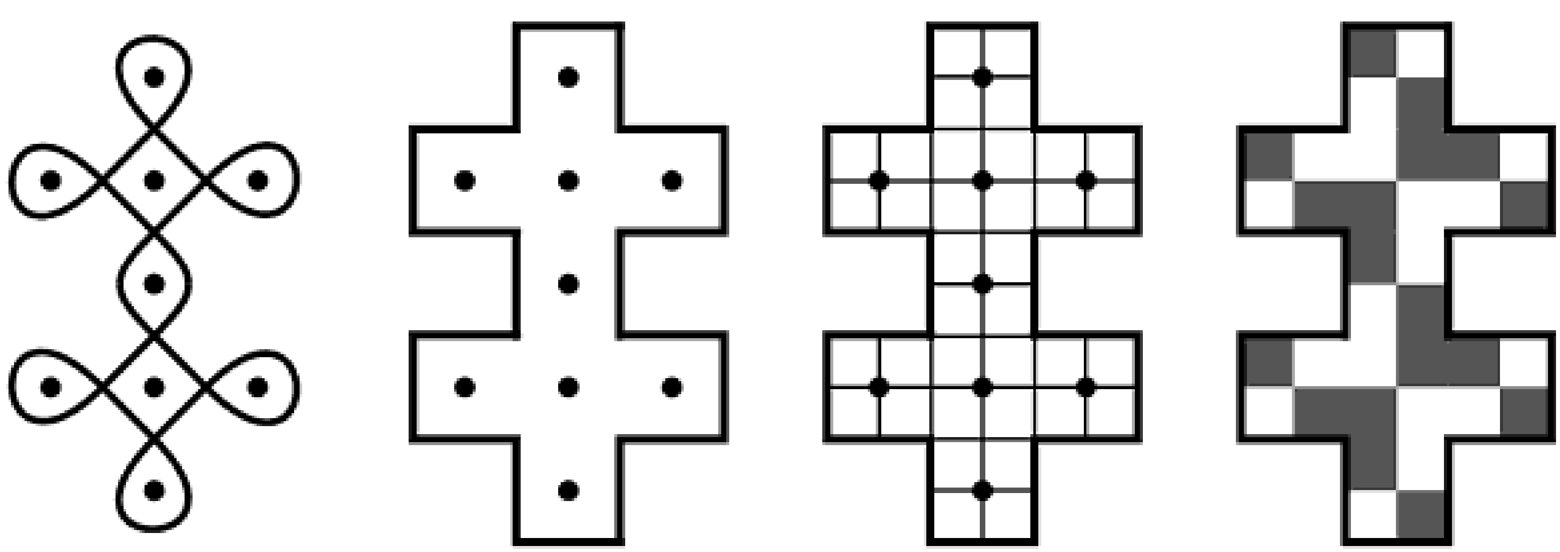
Examples of symmetrical 5x5 Lunda-designs together
with the circular Lunda-designs they generate
Figure 3.21



a b c d

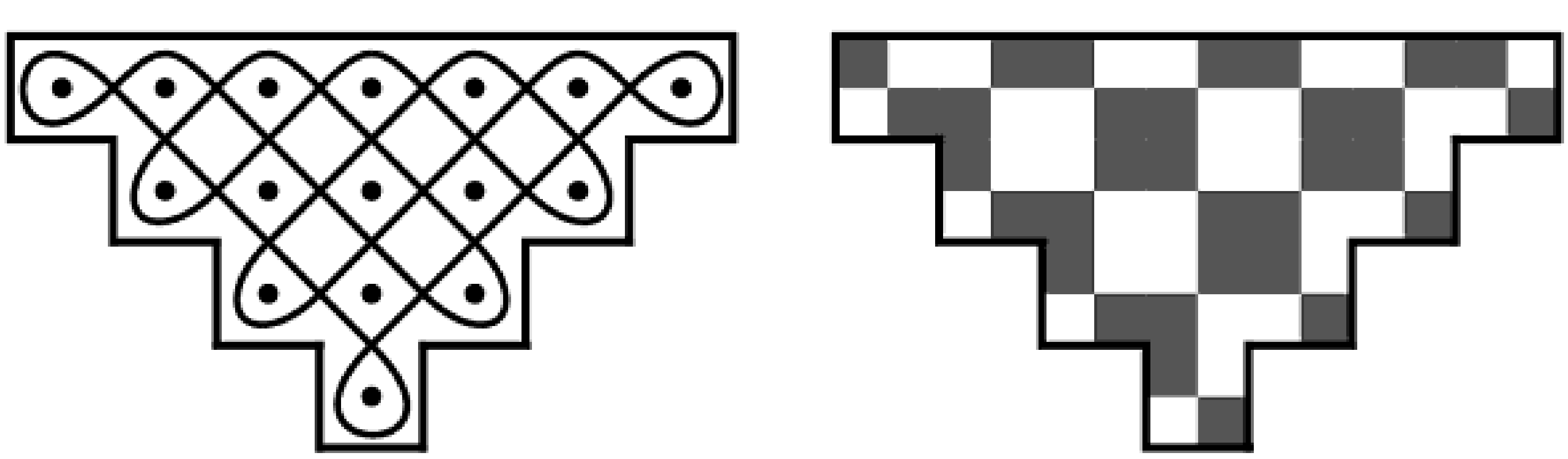
Example of the generation of a polyominal Lunda-design
Figure 3.22

Figure 3.23 displays an ancient Egyptian scarab decoration [Petrie, 1934, pl.VII, nº 220]; the polyomino in which it is inscribed; and the Lunda-design it generates. Figure 3.24 and 3.25 do the same for a Celtic knot design [Jones, pl.LXIV] and for another Tamil threshold design [Layard, p.137]. In the last case the artist also drew the polyominal border. The mirror curves in Figures 3.22, 3.23, 3.24, and 3.25 may be classified as *regular* in the above-discussed sense.



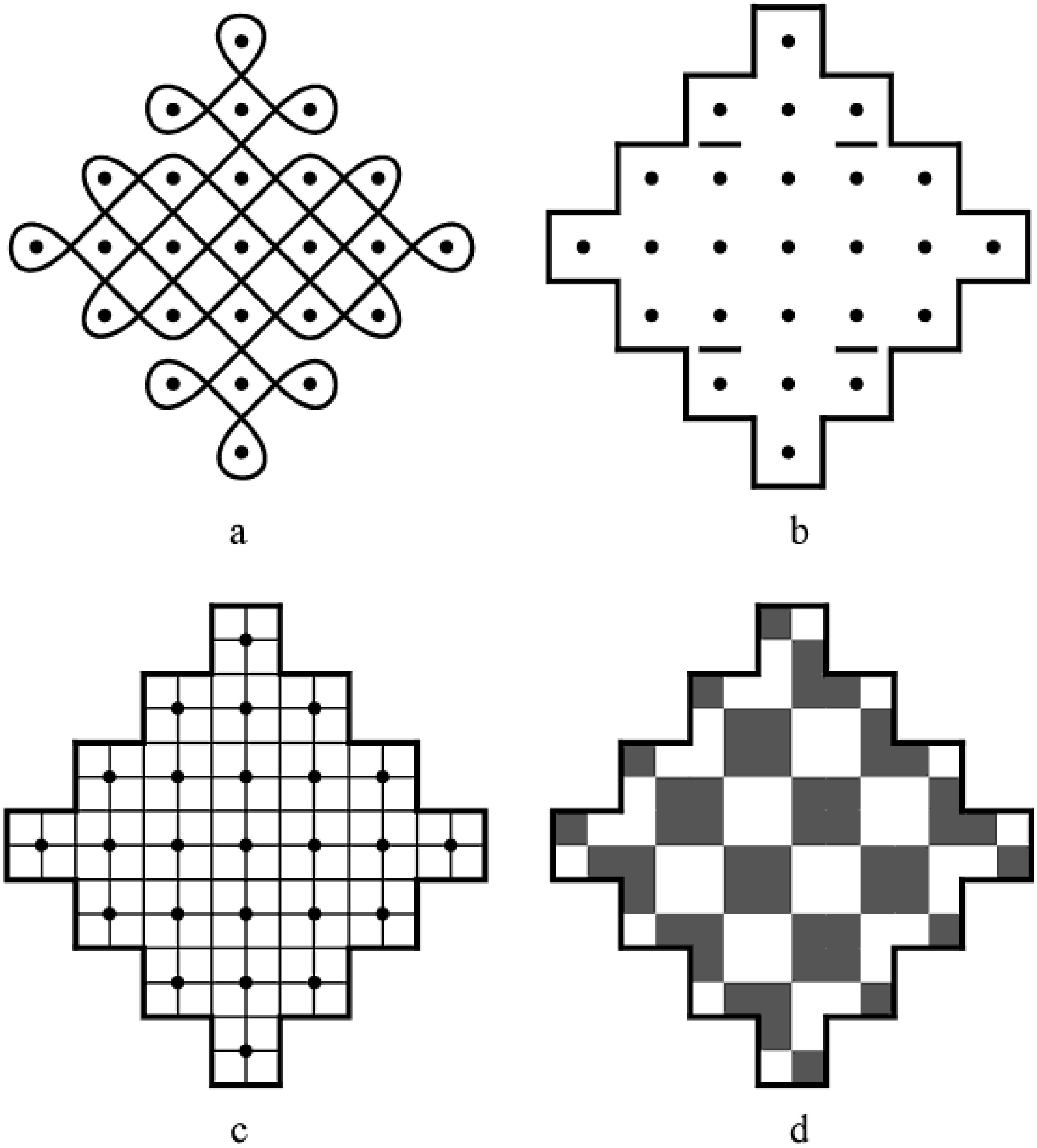
a b c d

Polyominal Lunda-design generated by an
ancient Egyptian scarab decoration
Figure 3.23



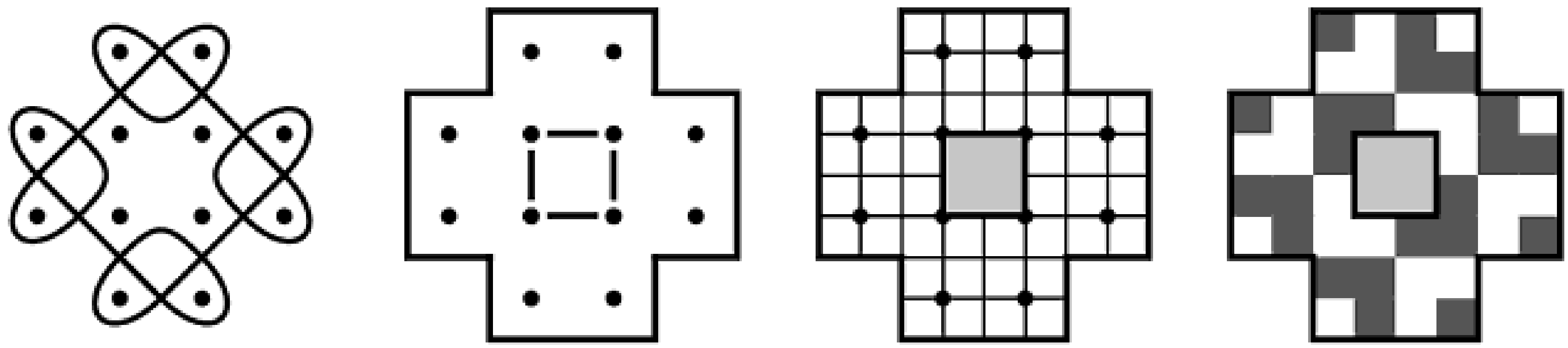
a b

Polyominal Lunda-design generated by a Celtic knot design
Figure 3.24



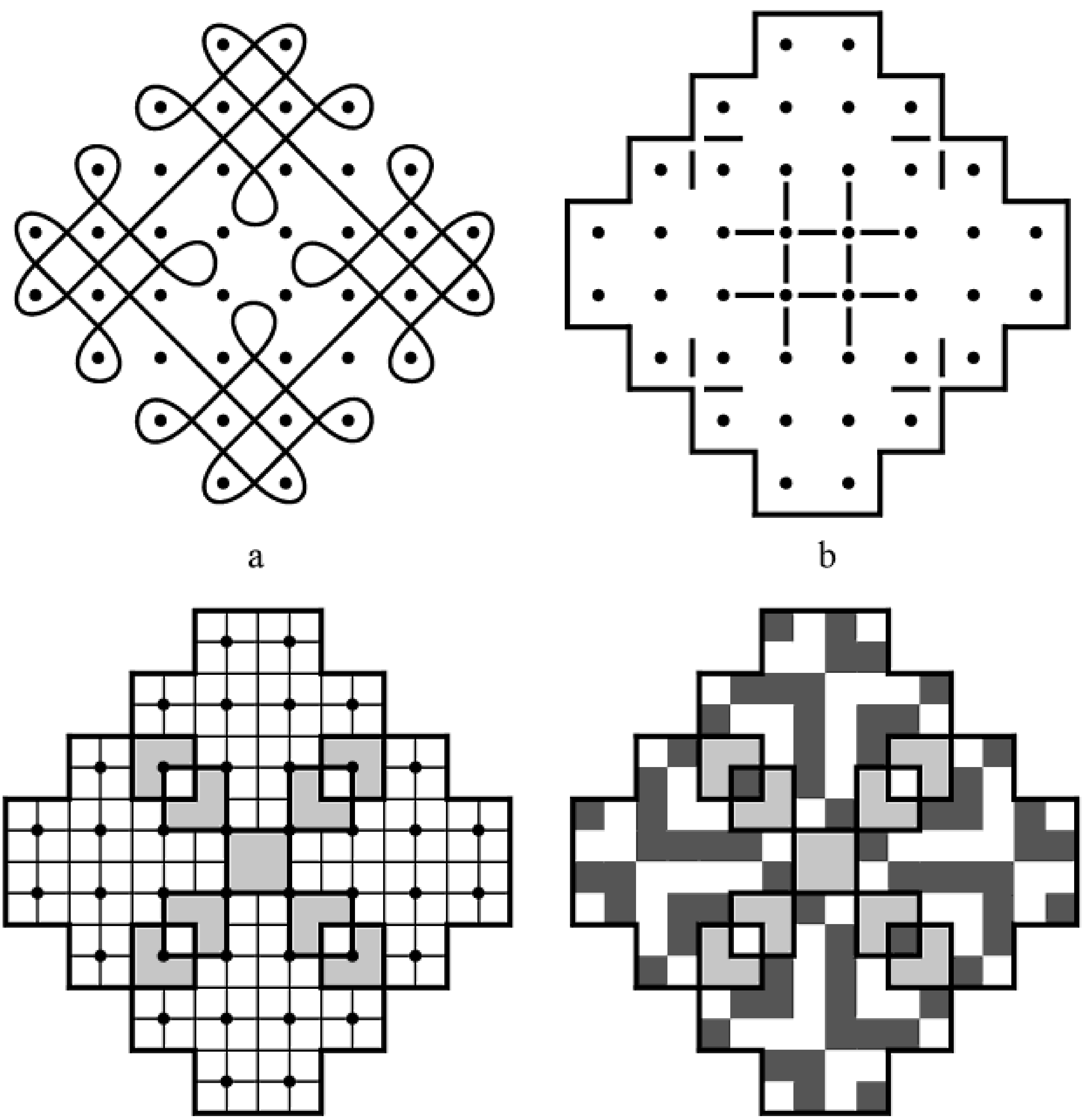
Polyominal Lunda-design generated by a Tamil threshold design
Figure 3.25

Figure 3.26a shows a Japanese crest design [Adachi, p.4; grid points added]. This time the mirror curve is not regular, as there are horizontal mirrors between horizontal neighboring grid points (see Figure 3.26b). By coloring the unit squares (see Figure 3.26c) through which the curve successively passes alternately black and white, the Lunda-design in Figure 3.26d is obtained. This time the curve does not pass through the central square. In other words we constructed a Lunda-design on a polyomino with a hole. A similar, though more complicated situation (see Figure 3.27) occurs in the case of an ancient Mesopotamian design (about 2800 B.C., cf. [Petrie, 1930, pl. XLI]; [Gerdes, 1993, chap.10]). In fact the curve does not fill the whole polygonal region, leaving several holes uncolored ('grey' in Figure 3.27c and d).



a b c d

Polyominal Lunda-design generated by a Japanese crest design
Figure 3.26



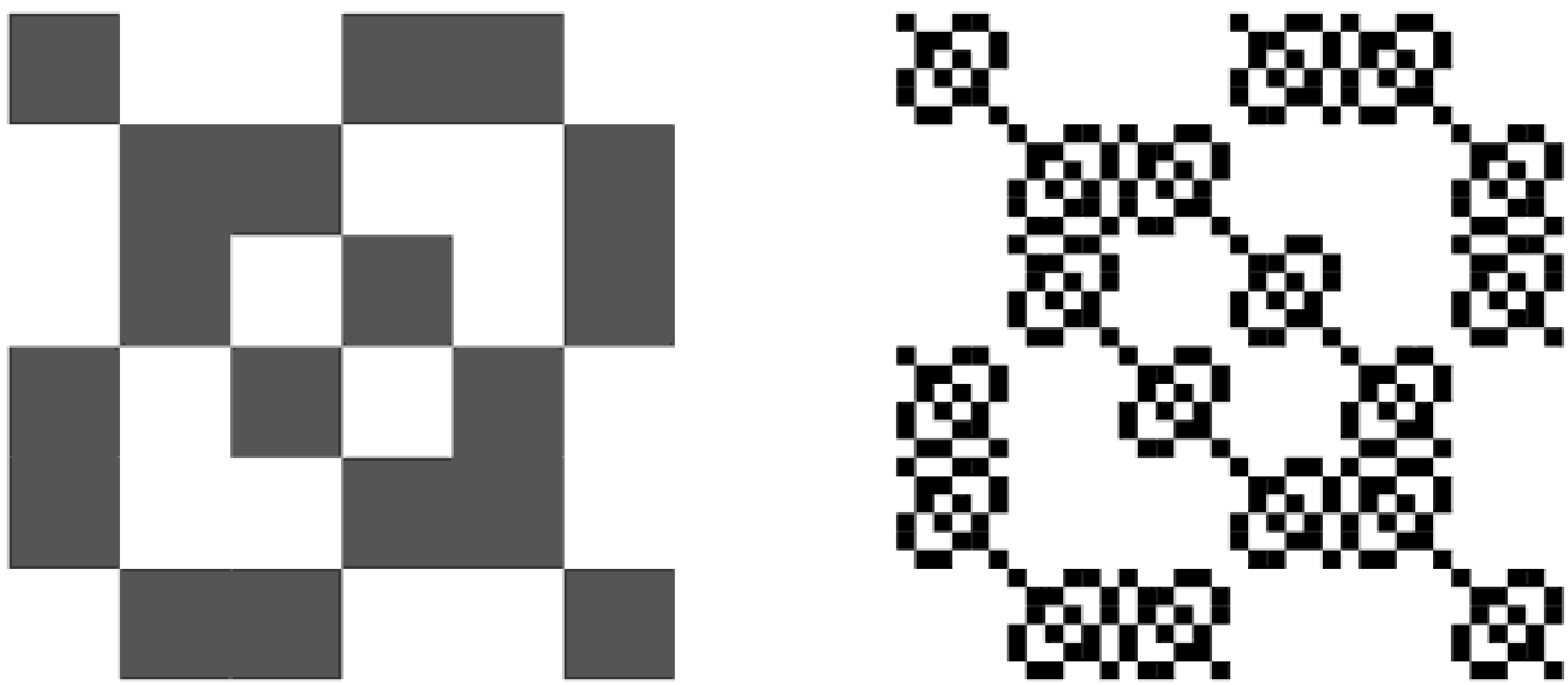
a b

c d

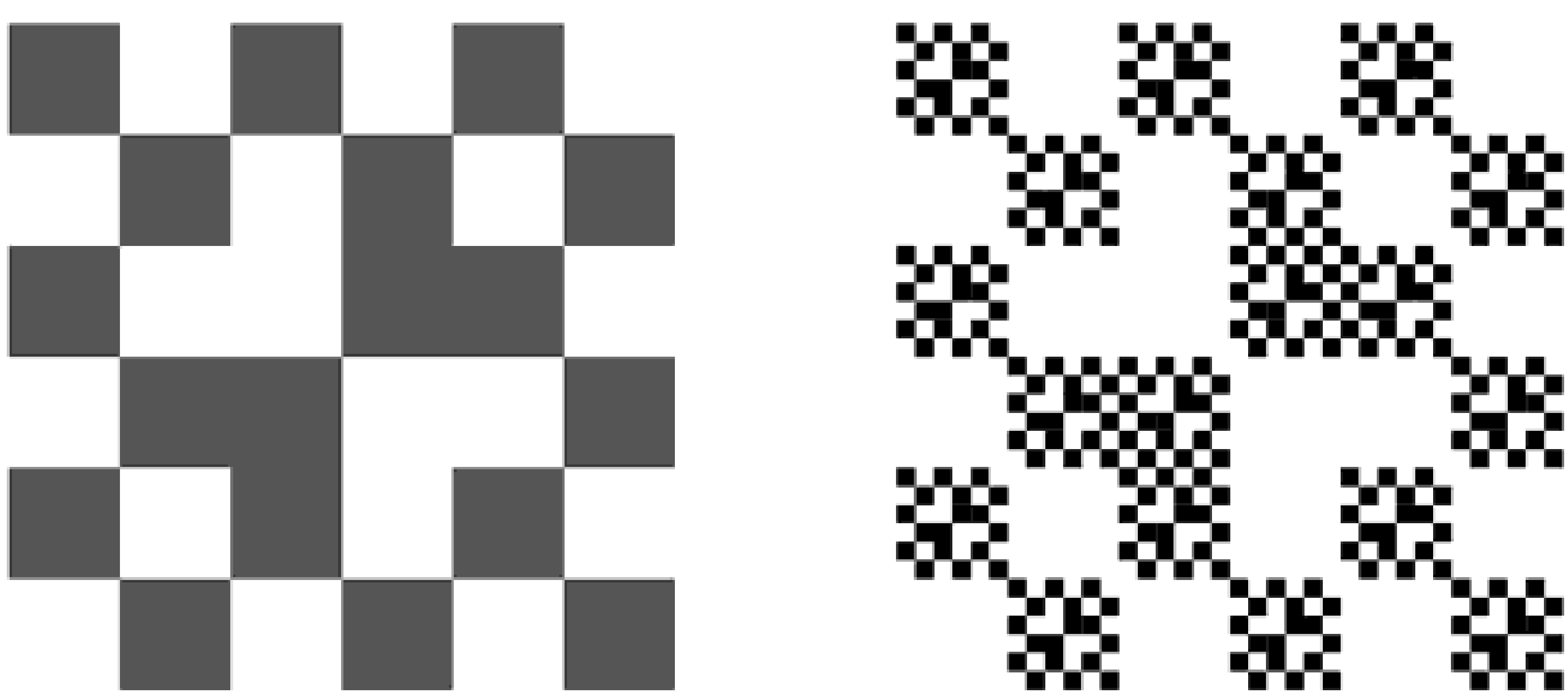
Polyominal Lunda-design generated
by an ancient Mesopotamian design
Figure 3.27

Fractal Lunda-designs

Square Lunda-designs may be used to build up fractals, that is geometrical figures with a built in self-similarity, by replacing each unit square with the original Lunda-design. Figure 3.28 shows the first two phases of the building up of two fractals on the base of 3x3 Lunda-designs.



a



b

First two phases of building up of two fractals
on the base of 3x3 Lunda-designs
Figure 3.28

The methods discussed in this paper for the generation of mirror curves and various types of Lunda-designs can easily be adapted to computer graphical representation.

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Chapter 4

ON LUNDA-DESIGNS AND LUNDA-ANIMALS

Fibonacci returns to Africa

4.1 Introduction

Lunda-designs are a certain type of black-and-white design. As they were discovered in the context of analyzing the properties of a class of curves (see the example in Figure 4.1) drawn in the sand among the Cokwe, who predominantly inhabit the north-eastern part of Angola, a region called Lunda, I have given them the name of **Lunda-designs** (cf. Gerdes, 1990; 1993-94, chapters 4 and 6; 1996a, b).

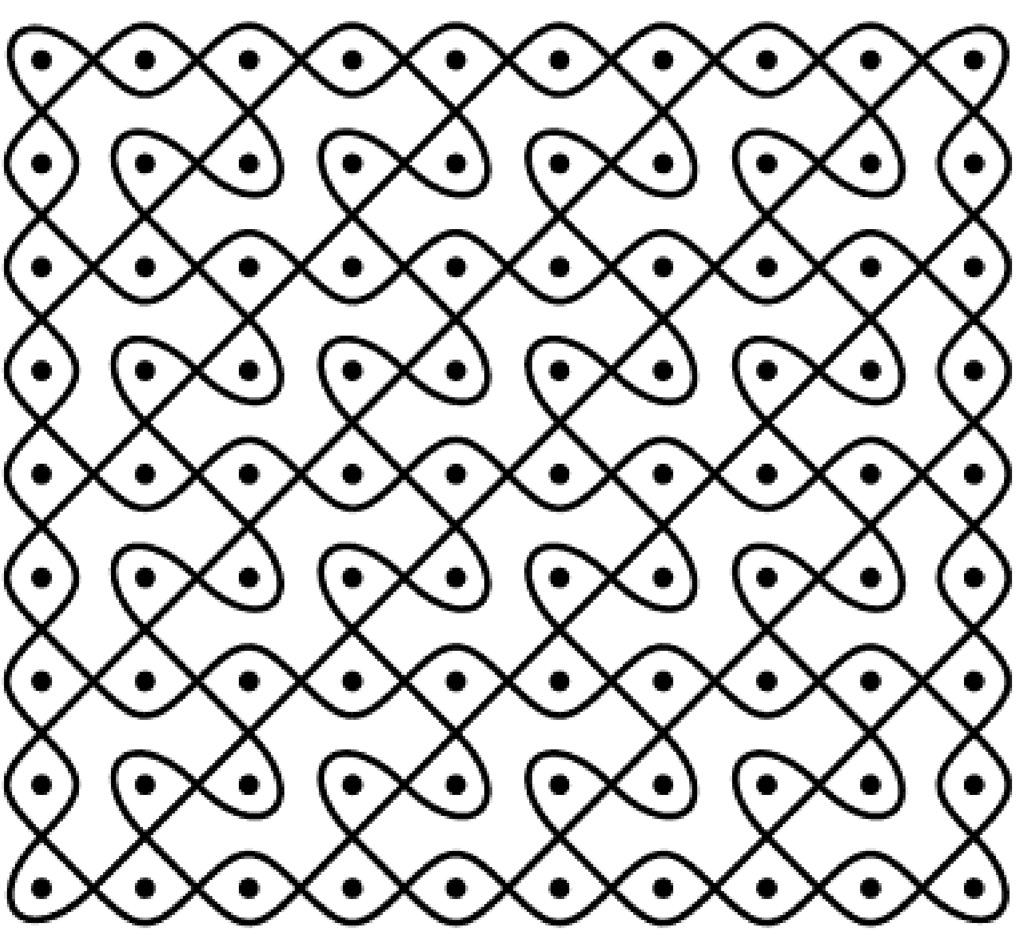


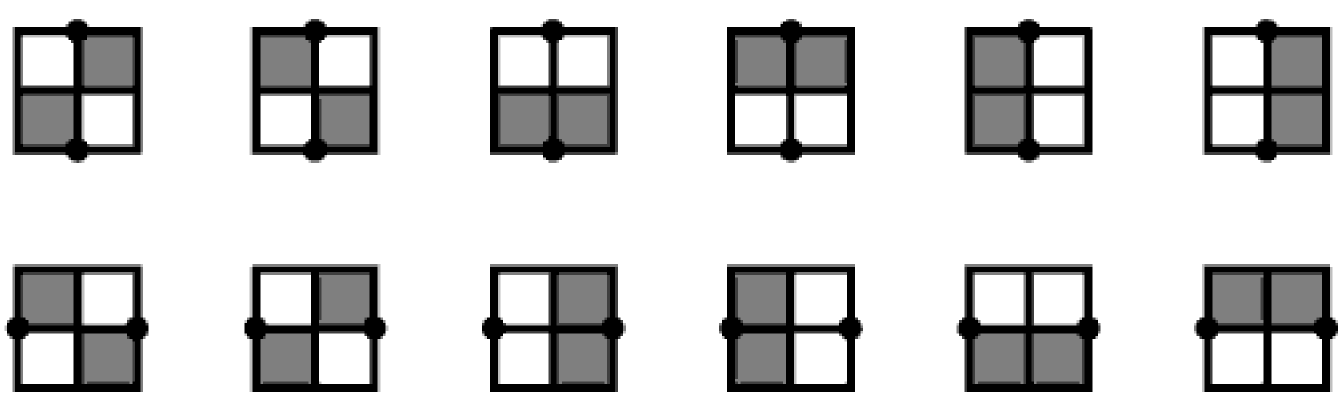
Figure 4.1

Consider an infinite grid having as points (2p, 2q), where *p* and *q* are two arbitrary whole numbers. An **infinite Lunda-design** may be defined as a black-and-white design with the following characteristic:

(i) Of the four unit squares (cells) between two arbitrary (vertical or horizontal) neighboring grid points, two are always black and two are white (see Figure 4.2a).

Figure 4.3 shows a finite Lunda-design. These finite Lunda-designs also have a second general property:

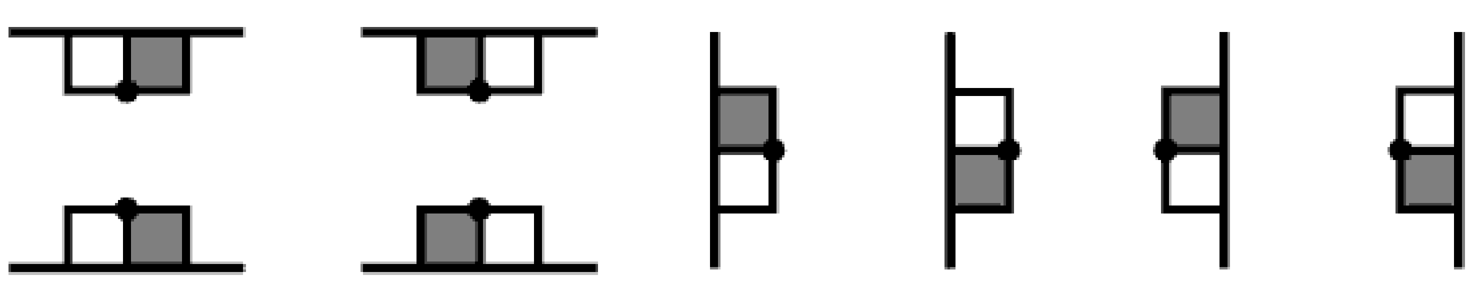
(ii) Of the two border unit squares of any grid point in the first or last row, or in the first or last column, one is always white and the other black (see Figure 4.2b).



Possible situations between vertical and horizontal neighboring grid points

a

Figure 2



Possible border situations

b

Figure 4.2

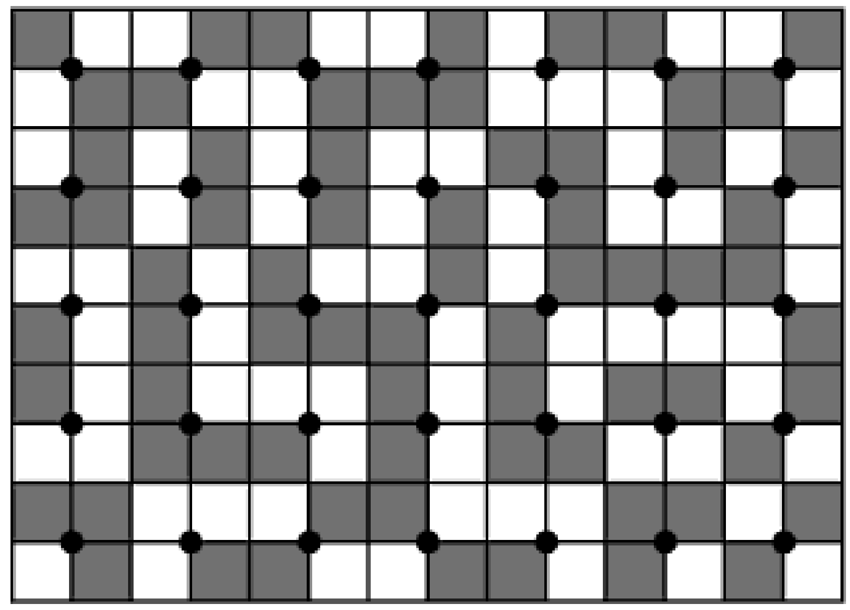


Figure 4.3

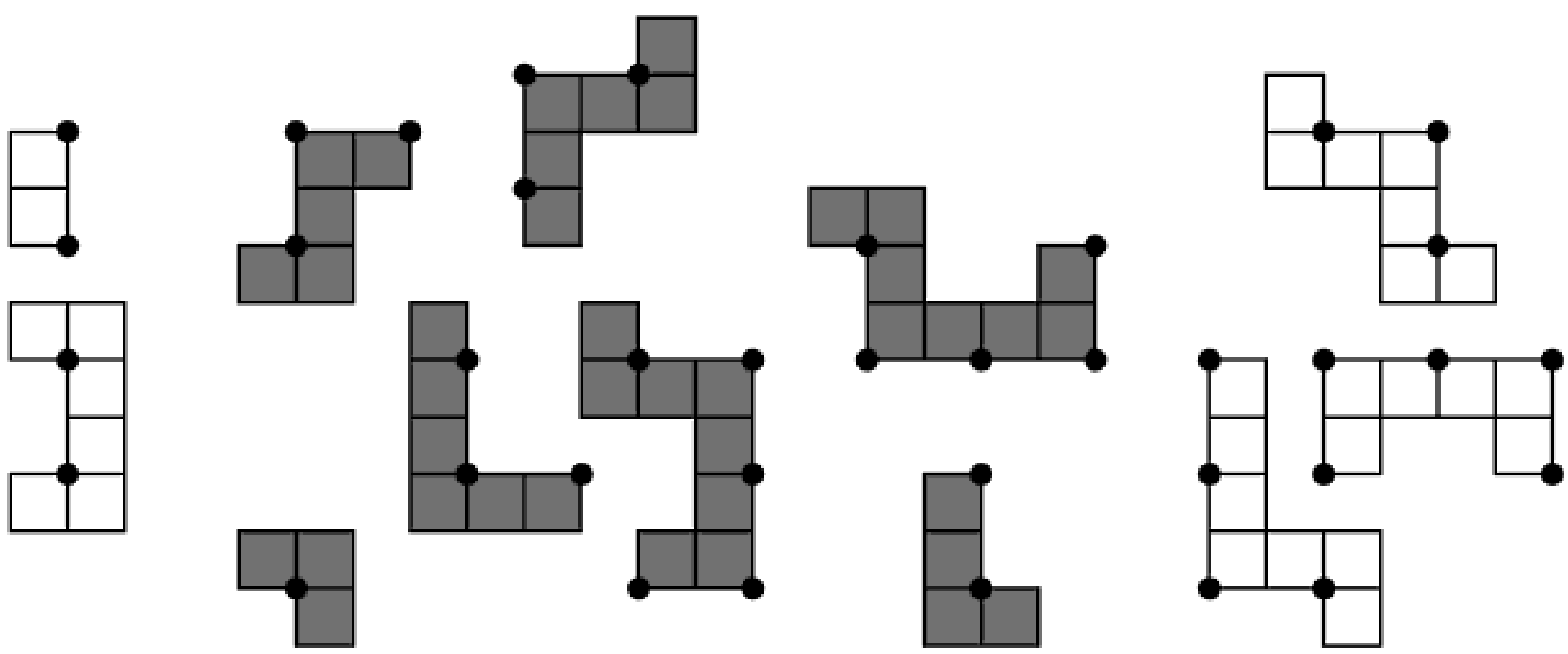


Figure 4.4

Polyominoes (either black or white) that appear in Lunda-designs will be called **Lunda-polyominoes**. Figure 4.4 displays some Lunda-polyominoes present in Figure 4.3.

In this chapter the number of possible paths of Lunda-animals will be analyzed. Here we define a **Lunda-animal** as a (black) Lunda-pentomino (consisting of 5 cells) with one unit square at one of its ends marked as head (H). A Lunda-animal walks in such a way that after each step the head occupies a new unit square, the second cell moves to the last unit square previously occupied by the head, the third cell to the unit square previously occupied by the second cell, etc. In other words, two subsequent positions of a Lunda-animal have a Lunda-tetromino in common. A path consists of the actual position of the Lunda-animal (black in the Figures) and all unit squares through which the animal passes (white in the Figures). A path may cross itself or repeat certain tracks (see the example in Figure 4.5).

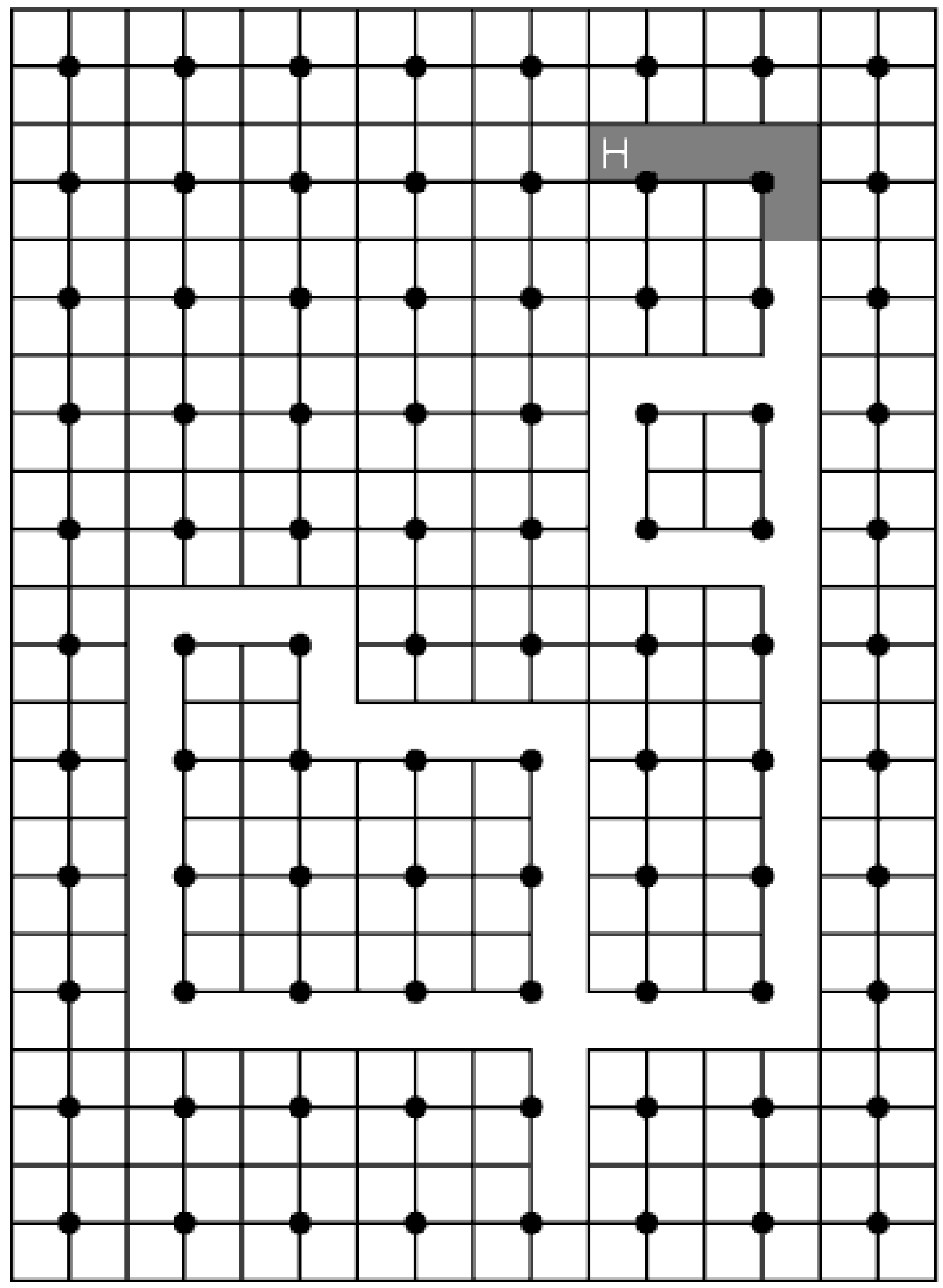


Figure 4.5

4.2 Counting the number of possible paths of Lunda-animals

Starting from the initial position in Figure 4.6a, how many paths p(n) of *n* steps are possible for a Lunda-animal?

There are three unit squares (marked 1, 2, 3 in Figure 4.6a) to which the head could move, in principle. To the third one, however, it cannot move as an inadmissible situation would emerge, with two neighboring grid points having three black unit squares between them in contradiction with characteristic (i) [see Figure 4.7]. Therefore, p(1) = 2; in the first case the animal 'bends its neck', in the second it keeps it straight (see Figure 4.6b).

From the 'bent neck' position, the animal's head can only proceed to the second neighboring unit square otherwise inadmissible situations would emerge. From the 'straight neck' position, the animal's head may once again proceed to two possible unit squares (2 and 3). In total, three paths of two steps are possible: p(2) = 3 (see Figure 4.6c).

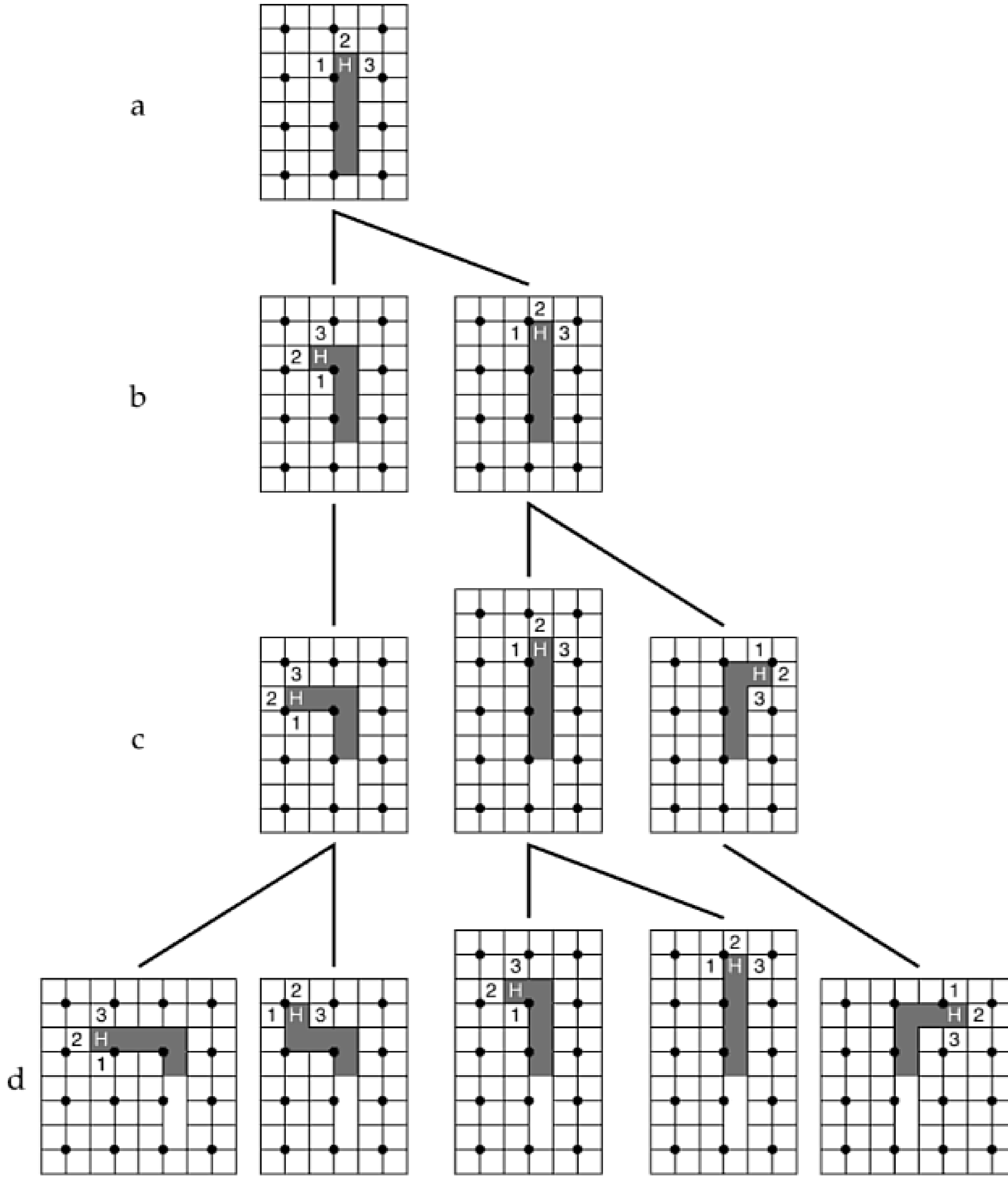


Figure 4.6

Of the three final positions of the Lunda-animal, two have straight necks leading each of them to two new positions in the next step, one with a bent neck and one with a straight neck. The third has a bent neck, giving rise to only one new (straight neck) position. In total five paths of three steps are possible: p(3) = 5, etc. (See Figure 4.6d, e, ...)

Let there be b(i) paths of *i* steps that end in a position with a bent neck, and s(i) with a straight neck. Each of these b(i) positions with a bent neck leads to one position with a straight neck after (i+1) steps.

Each of the s(i) positions with a straight neck leads, after one more step, to one position with a bent neck and one with a straight neck. In other words, for i=1, 2, ... we have:

(1) $b(i+1) = s(i)$, and
(2) $s(i+1) = b(i) + s(i) = p(i)$.

It follows that for n=2, 3,…, we have

$p(n+1) = b(n+1) + s(n+1) = s(n) + p(n) = p(n-1) + p(n)$, or
(3) $p(n+1) = p(n) + p(n-1)$.

The recurrence formula $f(n+1) = f(n) + f(n-1)$ with $f(1) = 0$, $f(2) = 1$ leads to the famous Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... [1]

As we have $p(1) = 2$ and $p(2) = 3$, our final result is:
(4) $p(n) = f(n+3)$ for n=1, 2, 3,...

If we had defined a **Lunda-animal** as a (black) Lunda-m-omino (consisting of *m* cells) with $m > 5$, the answer to the question of how many positions are possible after *n* steps is still the same $p(n) = f(n+3)$ for $m < 9$. From m=9 onwards, $p(n) < f(n+3)$ holds, as the following example illustrates. The only possible step after the position in Figure 4.7a (m=9) is for the head to go to unit square 3. According to characteristic (i) of Lunda-designs the head cannot go to unit square 1. And if it were to go to unit square 2 (see Figure 4.7b), five neighboring unit squares are white, which is also impossible according to characteristic (i) of Lunda-designs. In other words, although the position in Figure 4.7a was that of a straight neck, there is only one possibility for the animal to continue its path.

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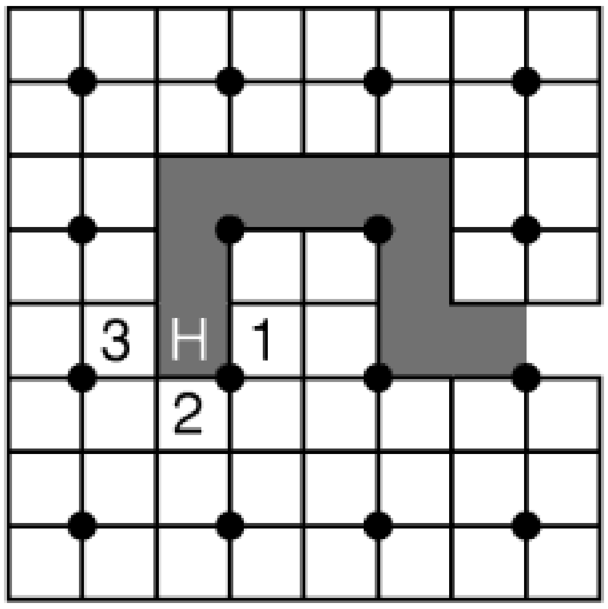
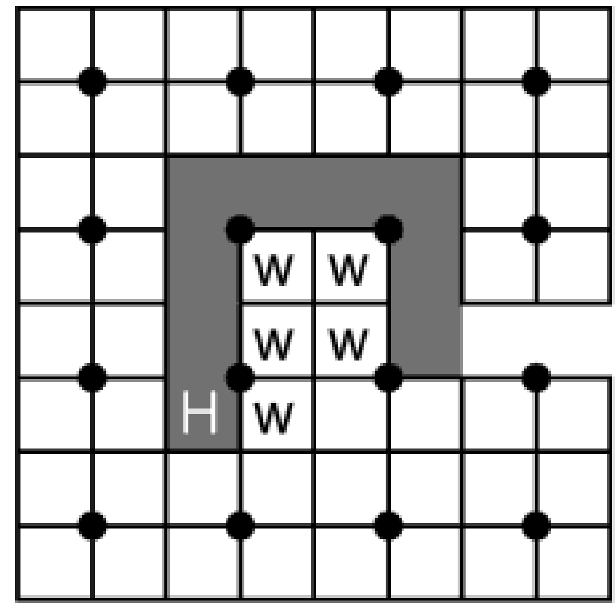
1 "As Fibonacci says himself, this Italian scholar was trained, when very young, in Bougie (in today's Algeria, p.g.), one of the Maghrebian scientific poles of the 12th century and, later, he reproduced, in his *Liber Abbaci*, certain aspects of the Maghrebian mathematical tradition" (Djebbar, 1995, 25). Probably he learnt about the 'Fibonacci' sequence when he was in North Africa.



a



b

Figure 4.7

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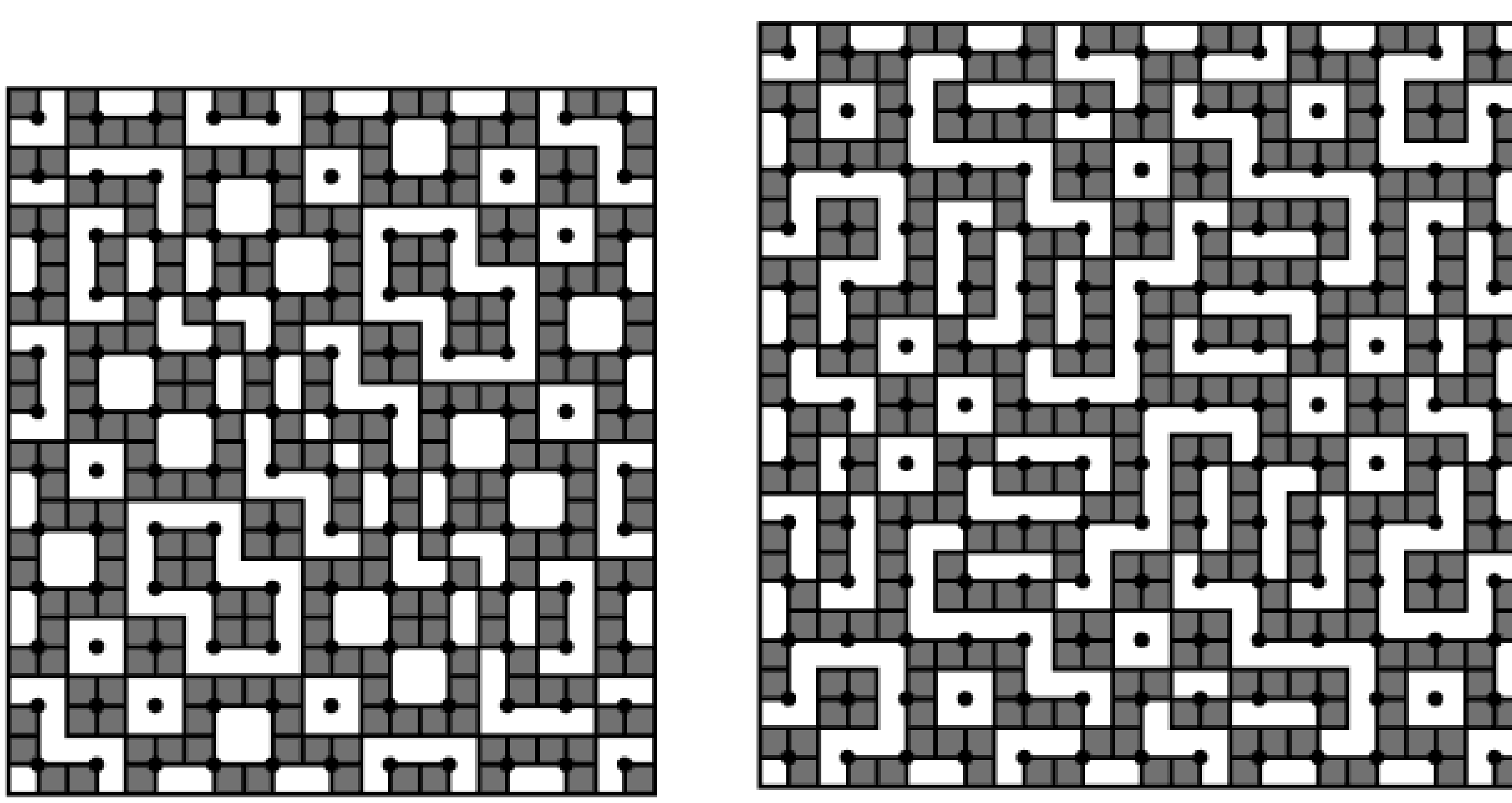
Chapter 5

ON LUNDA-DESIGNS AND POLYOMINOES

5.1 Polyominoes in Lunda-designs

Polyominoes (cf. Golomb) that appear in Lunda-designs will be called **Lunda-polyominoes**. Figure 5.2 displays some Lunda-polyominoes present in the Lunda-designs of Figure 5.1.

In this chapter the question of how many types of Lunda-n-ominoes there are, will be addressed. A first approximation of the number of Lunda-n-ominoes will be presented.



a b

Figure 5.1

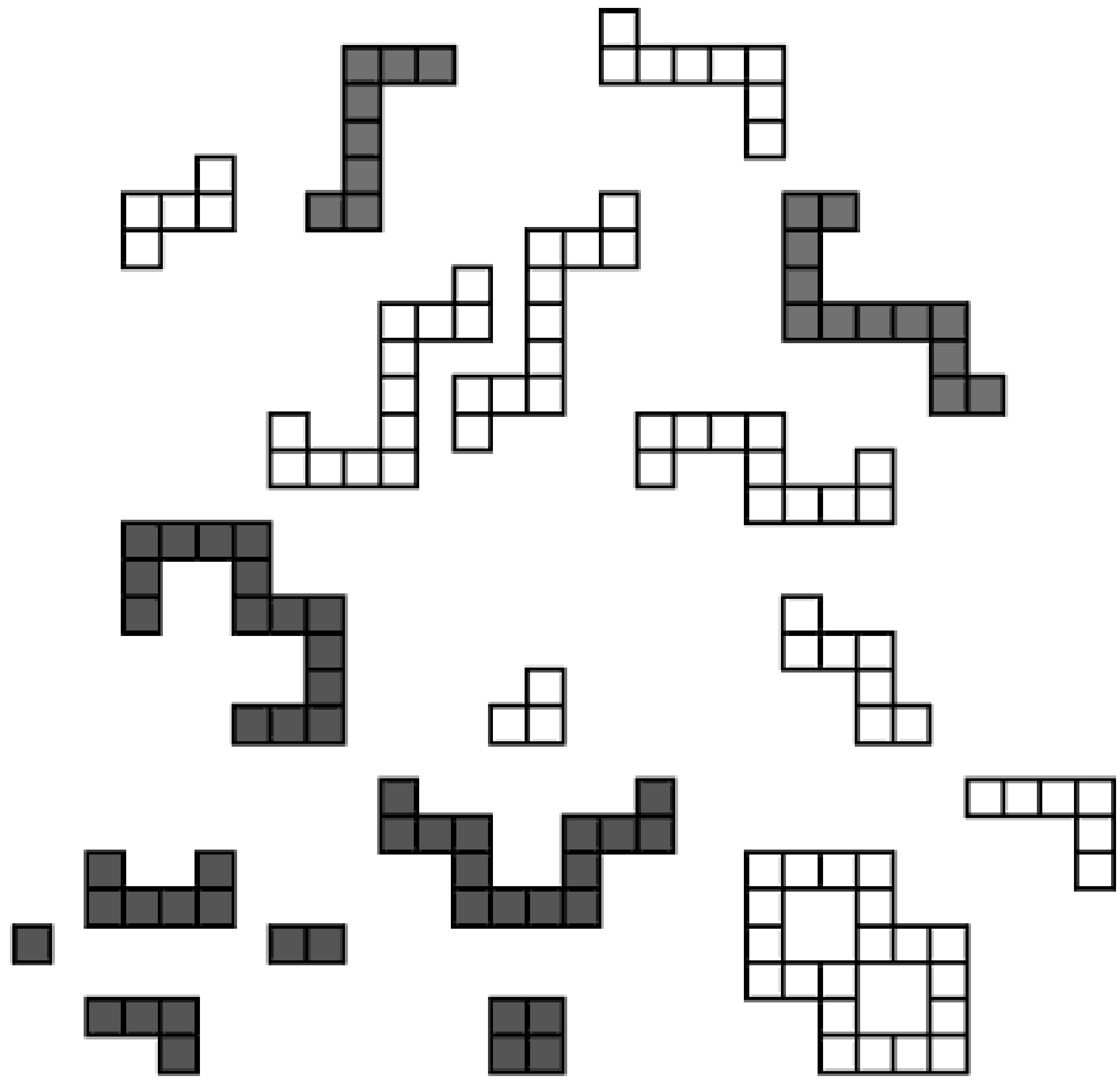


Figure 5.2

5.2 Types of Lunda-Polyominoes

For each n there is an infinite number of Lunda-n-ominoes. Various types of Lunda-n-ominoes are defined on this infinite set by notions of equivalence in terms of certain groups of isometries of the plane.

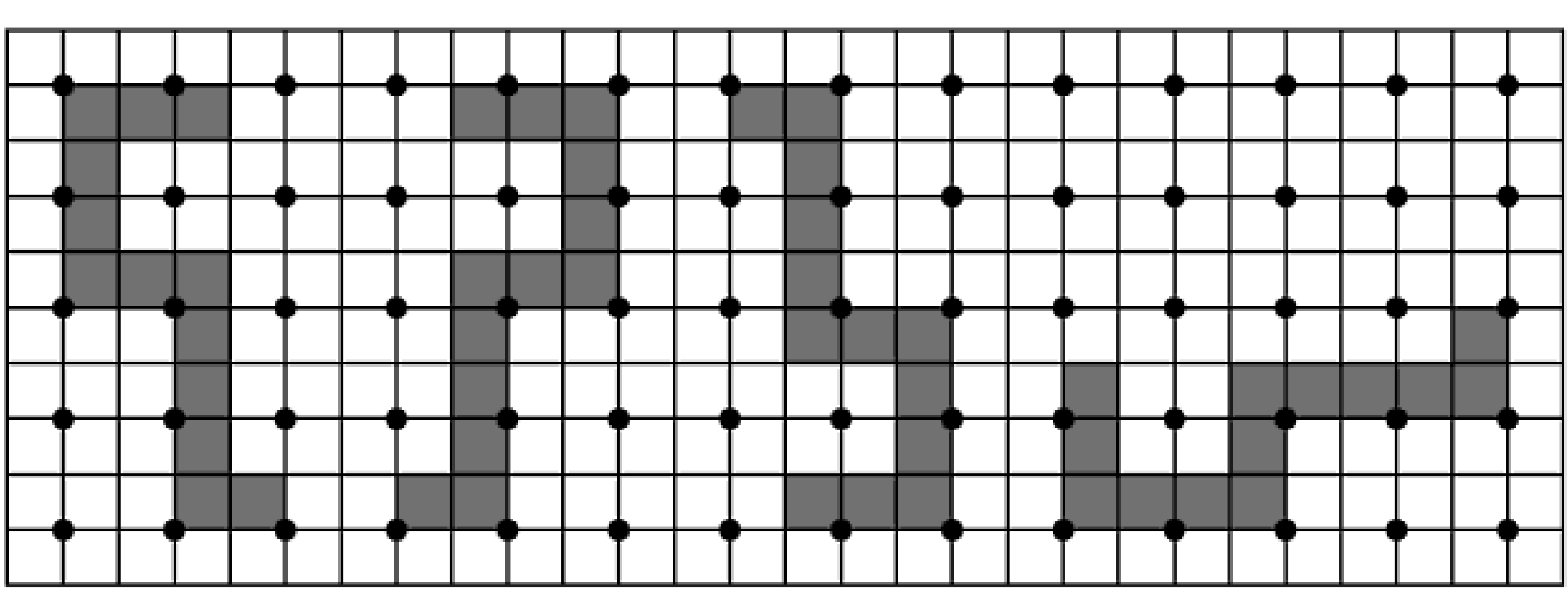


Figure 5.3

Figure 5.3 displays four Lunda-13-ominoes that are equivalent in the sense that for any of them there exists an appropriate translation, rotation or reflection, which maps it (together with its neighboring grid points) onto the first. What happens if one translates a Lunda-polyomino in such a way that its position relative to the grid changes essentially, as in the example in Figure 5.4? Is the second polyomino also a Lunda-polyomino?

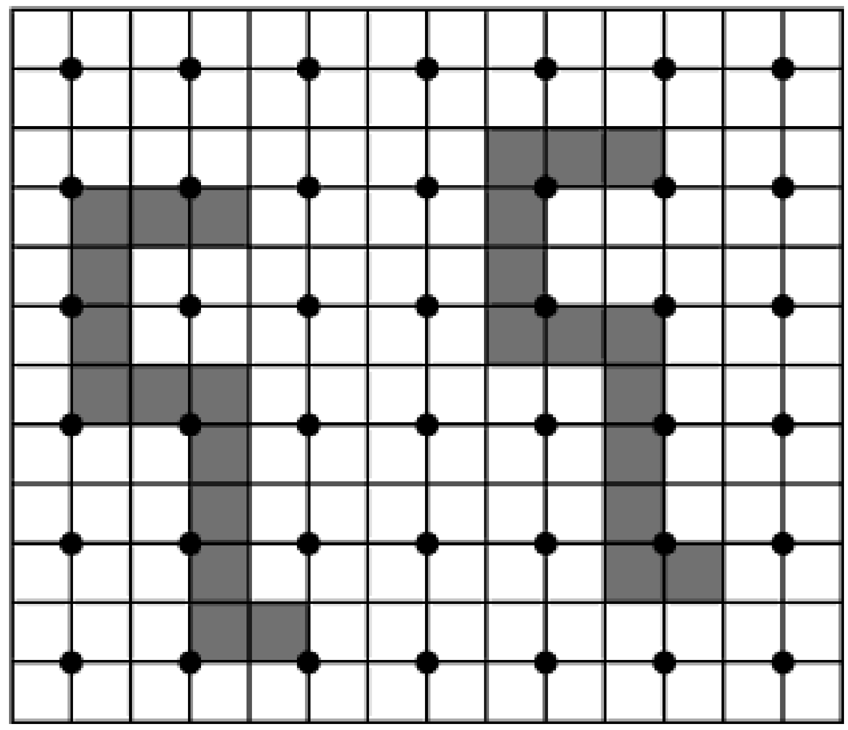
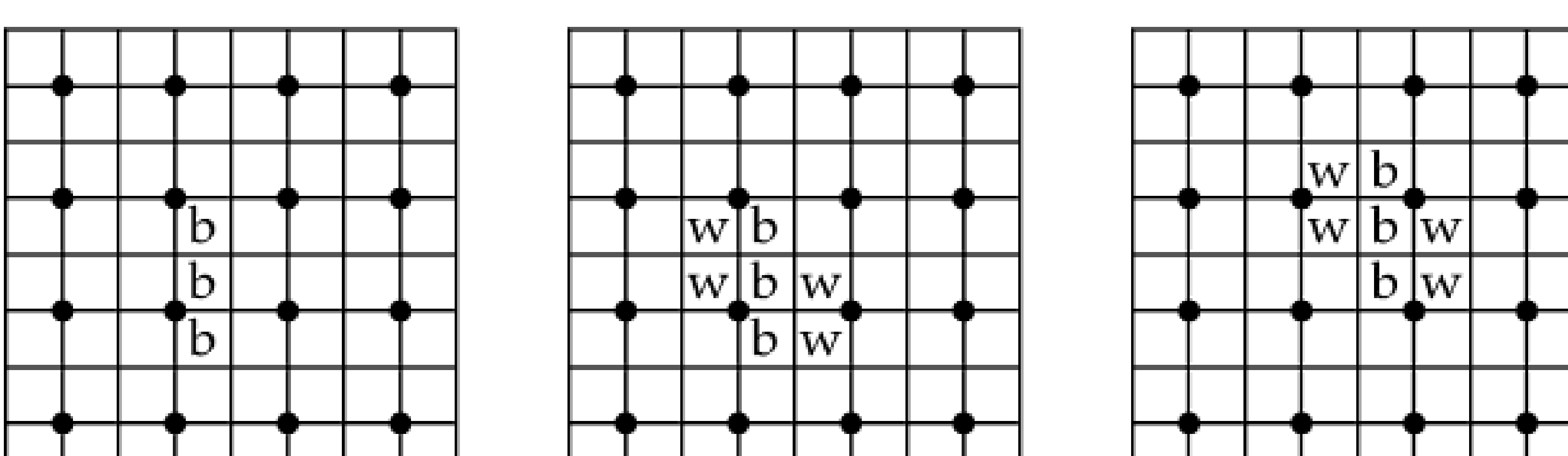


Figure 5.4

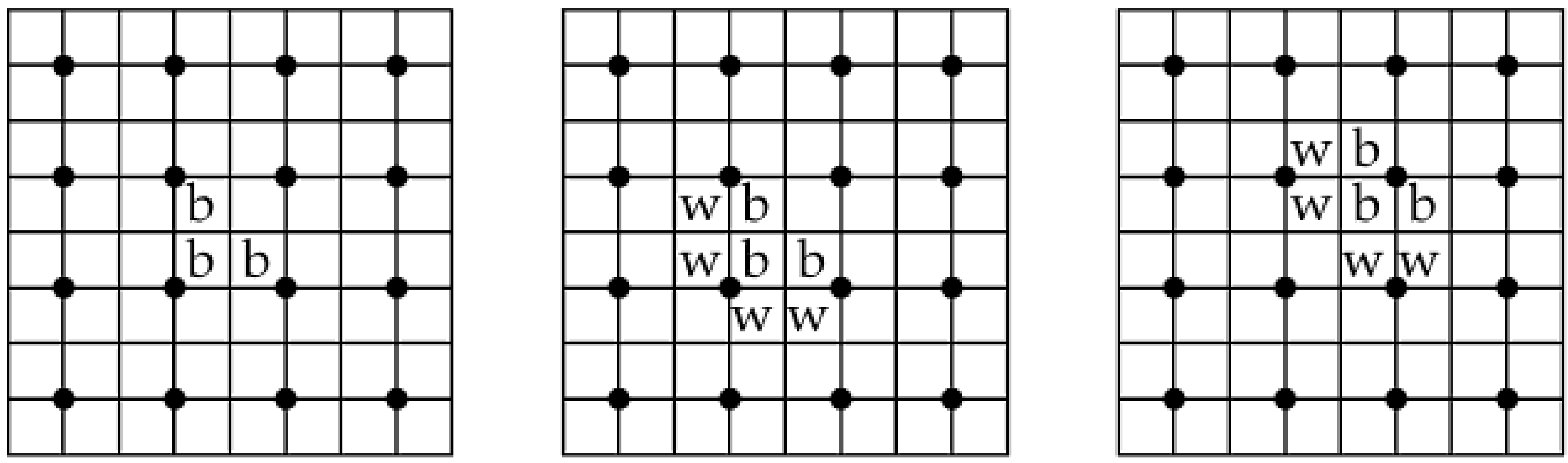
In the case of n=1, or n=2, the answer is immediately yes. Let us consider now a Lunda-n-omino with n > 2. It may be considered as composed of overlapping triominoes. The cells of each of the triominoes lie either in the same direction (as in Figure 5.5a, the cells are indicated by the letter b for black) or in a hook (as in Figure 5.6a). Figures 5.5b and 5.6b show which cells must be white in agreement with the second characteristic of Lunda-designs. When one translates these triominoes, together with their accompanying white cells, one unit to the right and then one unit upwards (see Figure 5.5c and 5.6c), it transpires that the resulting positions satisfy the second characteristic too. As this happens with all the successive overlapping triominoes, the same is true for the whole Lunda-n-omino under consideration.



a b c

Figure 5.5

In other words, in the example of Figure 5.4 we find that the second 13-omino is automatically a Lunda-13-omino that is equivalent to the first Lunda-13-omino.

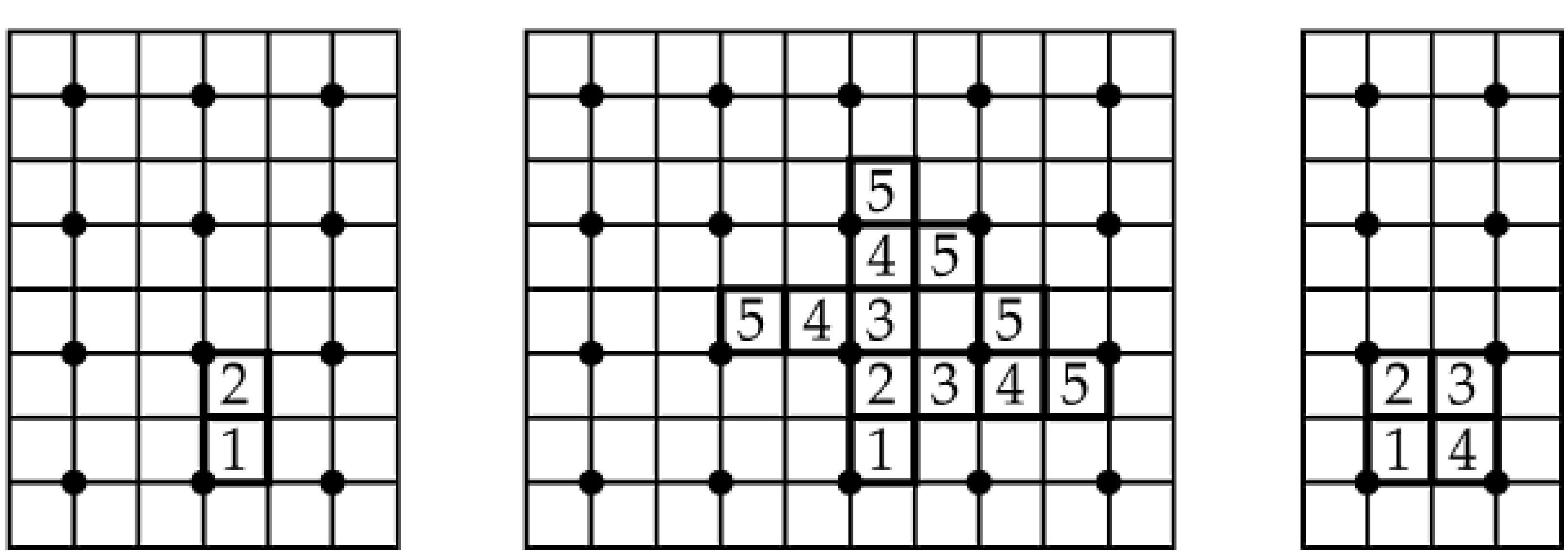


a b c

Figure 5.6

5.3. Approximation of the number of Lunda-n-ominoes

For each type of Lunda-n-omino (with $n \geq 2$) there exists a Lunda-n-omino that has its first two cells in a position like that of Figure 5.7a. For its third cell there are two possibilities (see Figure 5.7b), for its fourth cell there are three possibilities (if we do not count the abnormal case displayed in Figure 5.7c, where it is impossible to continue with a fifth cell, etc.), for its fifth cell five possibilities, etc. Therefore we find for the number a(n) of Lunda-n-ominoes with the given 'start position': $a(1) = 1$, $a(2) = 1$, $a(3) = 2$, $a(4) = 3$, $a(5) = 5$, the first terms of the famous Fibonacci sequence f(n) with $f(2) = f(1) = 1$ and $f(n+2) = f(n+1) + f(n)$. Will $a(n) = f(n)$ for all n?



a b c

Figure 5.7

If growth were to be unlimited — as in the case of Lunda-animals discussed in the previous chapter — we would have a(n) = f(n).

Figure 5.8 displays the corresponding types of Lunda-n-ominoes for n=1,...,7.

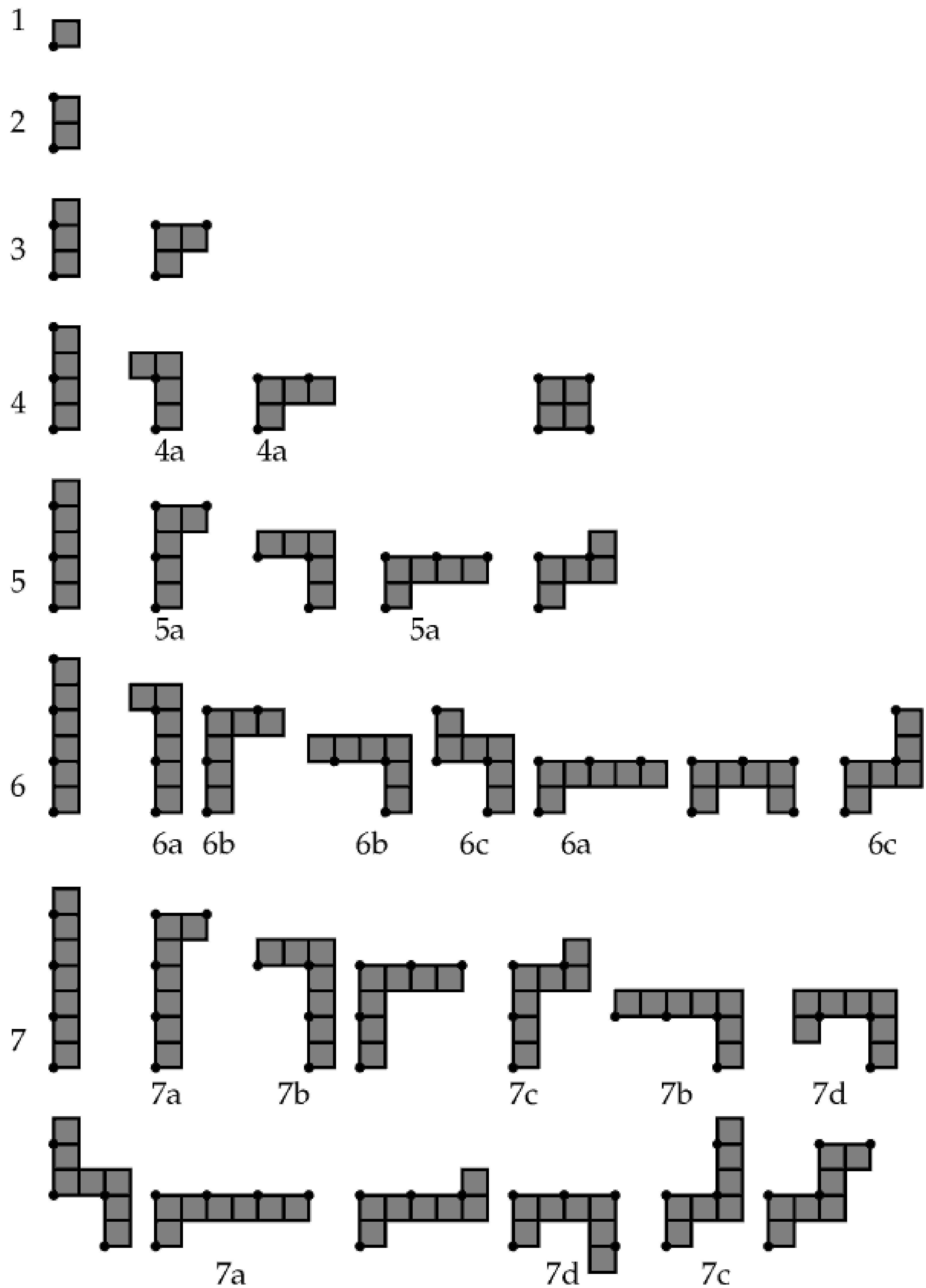


Figure 5.8

Types that are not symmetrical naturally appear twice in the list (Figure 5.8), as they may be obtained starting from either end. Therefore we have, in principle (if unlimited growth is permitted), for the number b(n) of types of Lunda-n-ominoes:

(4) $b(n) = [s(n) + a(n)] / 2$, or $b(n) = [s(n) + f(n)] / 2$,

where s(n) denotes the number of symmetrical types of Lunda-n-ominoes.

5.4 Symmetrical types

To evaluate s(n), consider odd and even *n* separately.

Case: *n* is odd

Figure 5.9 displays the symmetrical types for n=1, 3, ... , 9. Let n=2m-1.

i) Any Lunda-(2m-1)-omino that is invariant under a half-turn may be obtained by joining a Lunda-m-omino with the given 'start position' to its copy rotated through an angle of 180° around the centre of its first unit cell (see the example in Figure 5.10).

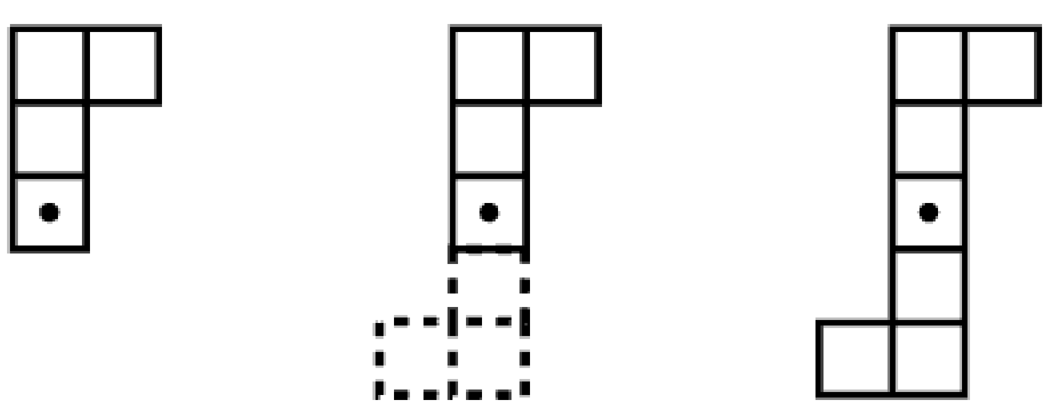
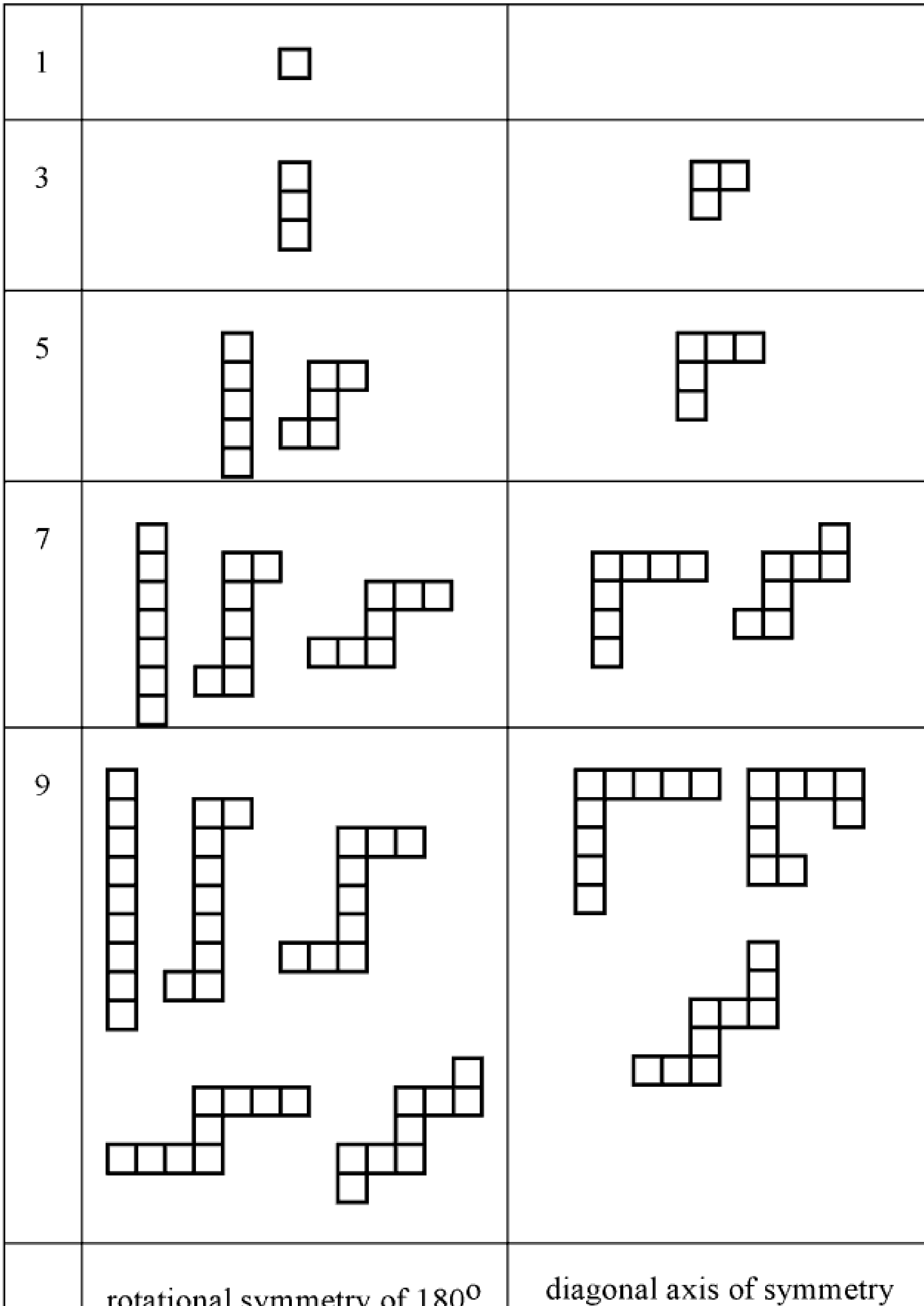


Figure 5.10

Therefore there are, at most, a(m)=f(m) Lunda-(2m-1)-ominoes with rotational symmetry of order 2.



Symmetrical types of odd order
Figure 5.9

ii) An odd symmetrical Lunda-n-omino may have a horizontal or vertical axis of symmetry only if it is a straight segment, as the distance between two hooks curved in the same direction has always to be an even number of unit cells (see Figure 5.11). As the straight segment n-minoes have already been taken into acount (i), they do not need to be considered again. All the other Lunda-n-ominoes of odd order which are invariant under a reflection, may be obtained from the hook triomino by joining, in a symmetrical way and in agreement with the second characteristic of Lunda-designs, the same number of unit cells at its two ends. As the number of ways to join k unit cells at each end of the hook triomino is at most f(k+1) (if unlimited growth is permitted), there are at most f(k+1) types of Lunda-(2k+3)-ominoes with a diagonal axis of symmetry.
From 2k+3 = 2m-1, it follows that there are at most f(m-1) types of Lunda-(2m-1)-ominoes with a diagonal axis of symmetry.

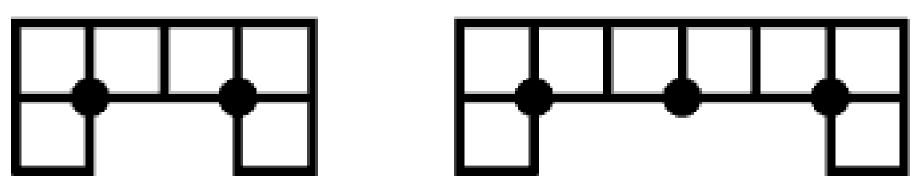


Figure 5.11

In agreement with (i) and (ii), there are, at most, f(m)+f(m-1), that is f(m+1), symmetrical types of Lunda-(2m-1)-ominoes. Thus:

(5) $b(2m-1) \leq [s(2m-1) + f(2m-1)]/2 \leq [f(m+1) + f(2m-1)]/2.$

Case: n is even

Let n=2m.

i) Suppose that open Lunda-2m-ominoes with only a rotational symmetry of order 2, exist. Then the centre of rotation can only be the midpoint of the common edge of the m^{th} and $(m+1)^{th}$ unit cells. As the image, under a half turn about this point, of the last hook before the mth unit cell — both at an even or both at an odd distance from the midpoint (see Figure 5.12) — is impossible, according to the second characteristic of Lunda-designs, open Lunda-2m-ominoes with only a rotational symmetry of order 2 do not exist.

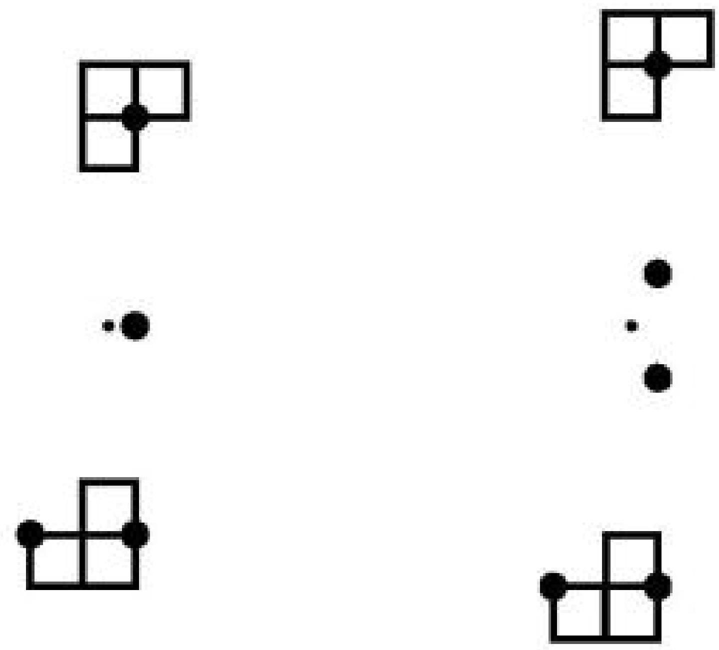
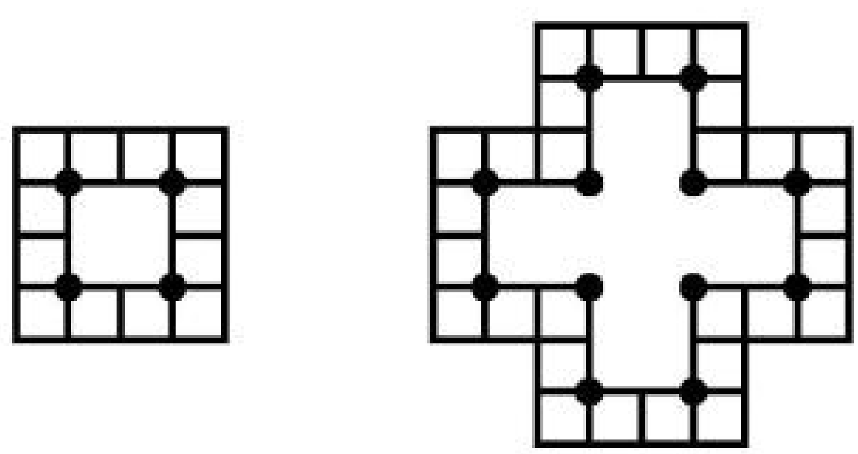


Figure 5.12

Open in this context means that the first and last unit squares do not have a common edge; if this happens we call the Lunda-polyomino **closed** (see the examples in Figure 5.13).



Examples of closed Lunda-polyominoes (n = 12, 28)
Figure 5.13

In rather exceptional cases closed Lunda-polyominoes with only rotational symmetry of order 2 may exist. The first two appear for n=36 (see Figure 5.14).

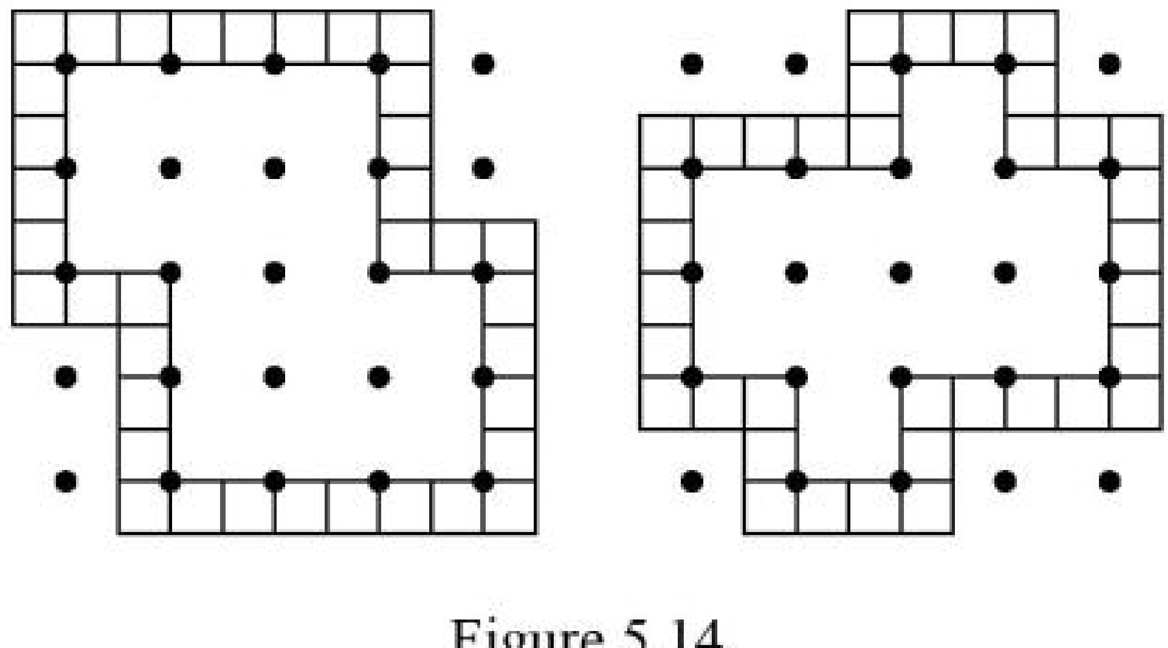


Figure 5.14

ii) Types of Lunda-2m-ominoes, which are invariant under a reflection, may be obtained from types of Lunda-m-ominoes by reflecting them in certain horizontal or vertical mirror lines.

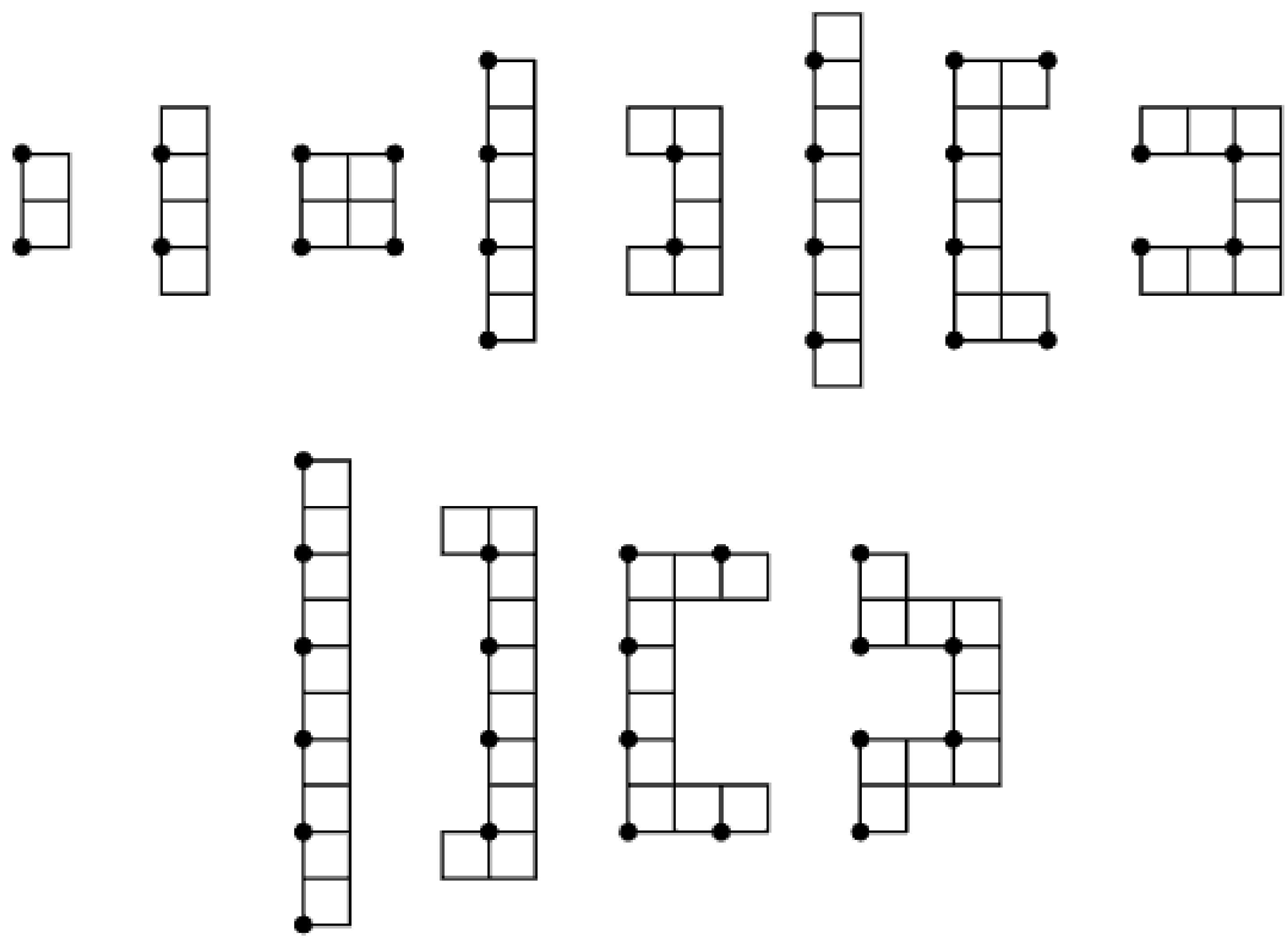
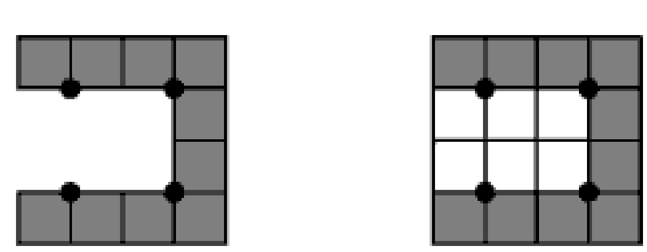


Figure 5.15

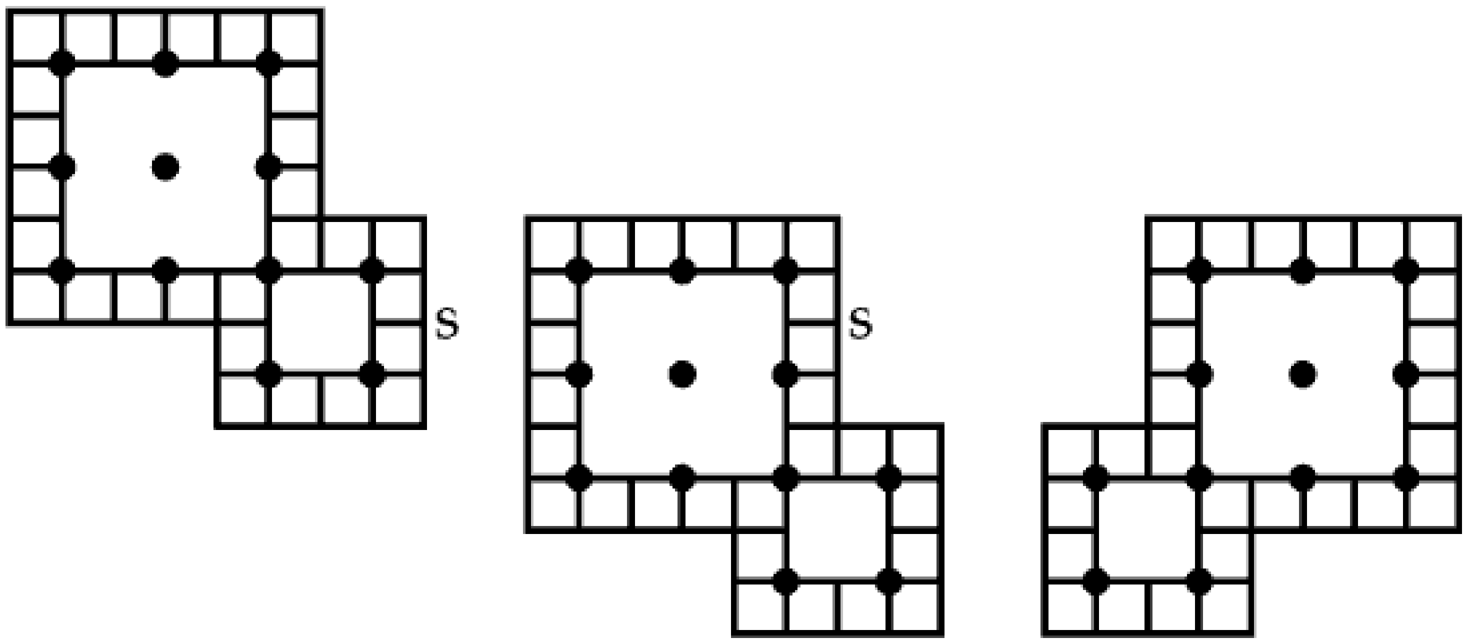
Figure 5.15 displays the symmetrical types for n=2, ... , 10. Not all of them are admissible. The black Lunda-10-omino presented in Figure 5.16a is not admitted, as it presupposes the existence of the white Lunda-polyomino in Figure 5.16b, which cannot exist as consequence of the second characteristic of Lunda-designs. Closed symmetrical types with the given start position (s) may appear several times, as the example in Figure 5.17 shows.



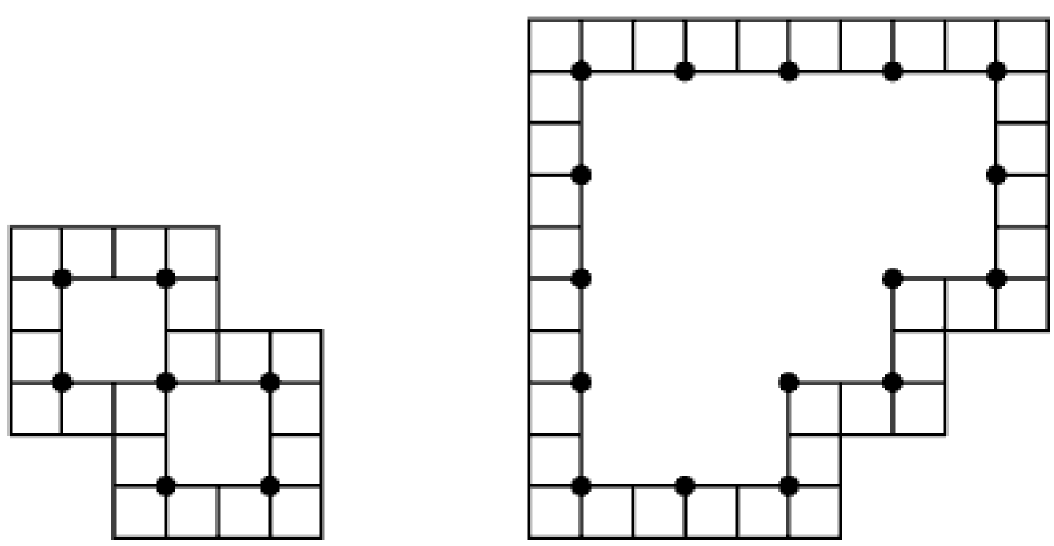
a b

Figure 5.16

For m = 6+4k (k = 1, 2, ...) closed Lunda-2m-ominoes exist with only a diagonal axis of symmetry (see the examples in Figures 5.17 and 5.18).



Three equivalent closed Lunda-28-ominoes
with different ‘start positions’
Figure 5.17



Examples of closed Lunda-2m-ominoes
with only a diagonal axis of symmetry (m= 10, 18)
Figure 5.18

In agreement with (i) and (ii), we have that the number s(2m) of types of symmetrical Lunda-2m-ominoes is at most equal to a(m) = f(m) for m>2.

It follows that

(6) $b(2m) \leq [s(2m) + f(2m)]/2 \leq [f(m) + f(2m)]/2$ for m>2.

We may conclude that the function c(n) with

$$c(n) = [f(m) + f(2m-1)]/2, \text{ if } n = 2m-1;$$

and

$$c(n) = [f(m) + f(2m)]/2, \text{ if } n=2m,$$

where *m* denotes a natural number, is a first approximation for the total number b(n) of types of Lunda-n-ominoes.

Table 5.1 presents the values of b(n) and c(n) for n=1, ..., 10.

Table 5.1

n	**f(n)**	**c(n)**	**b(n)**	**c(n)-b(n)**
1	1	1	1	0
2	1	1	1	0
3	2	2	2	0
4	5	2	3	-1
5	8	4	4	0
6	13	5	5	0
7	21	9	9	0
8	34	12	12	0
9	55	21	20	1
10	89	30	26	4

Figure 5.19 displays the four inadmissible 10-ominoes.

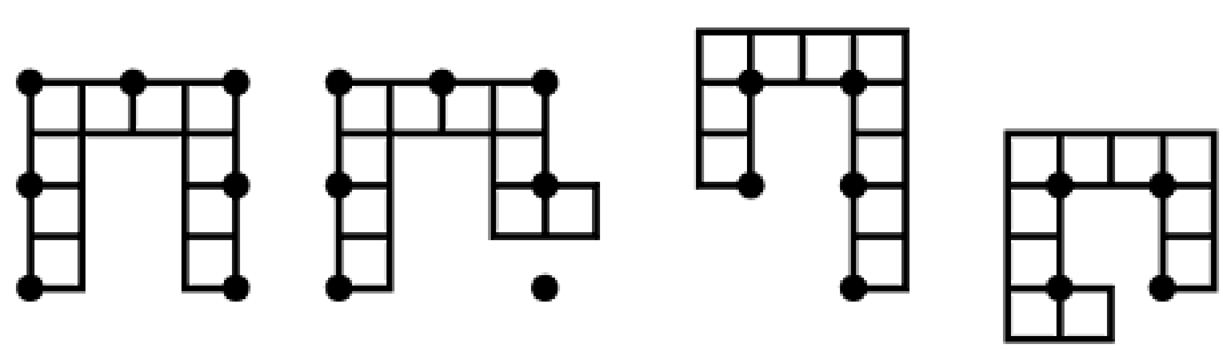


Figure 5.19

References

Golomb, Solomon W. (1994), *Polyominoes: Puzzles, Patterns, Problems, and Packings*, Princeton NJ: Princeton University Press, xii + 184 pp.

Chapter 6

SYMMETRICAL, CLOSED LUNDA-POLYOMINOES

In this chapter we will present some attractive examples of polyominoes with holes, which are Lunda-polyominoes in the sense that they may appear in Lunda-designs.

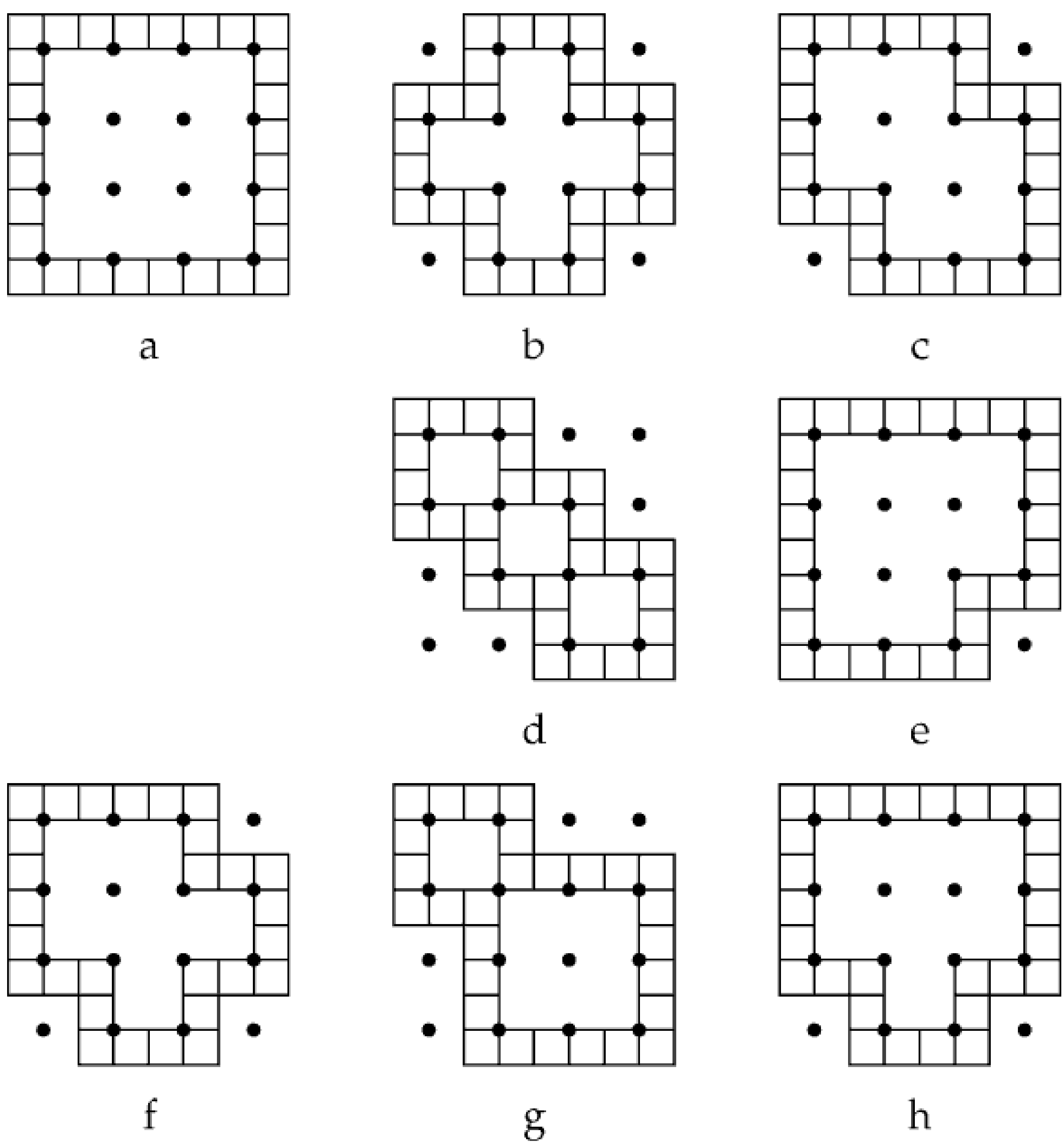


Figure 6.1

Figure 6.1 presents the eight types of symmetrical closed 28-ominoes, which are Lunda-polyominoes. Figure 6.2 shows that their interiors may be filled in by coloring the unit squares either black or white in such a way that the interiors together with the circumscribed polyominoes may be part of a Lunda-design. Mostly only one coloring is possible. In the case of the third 28-omino there are two possibilities, as displayed in Figure 6.2c1 and c2.

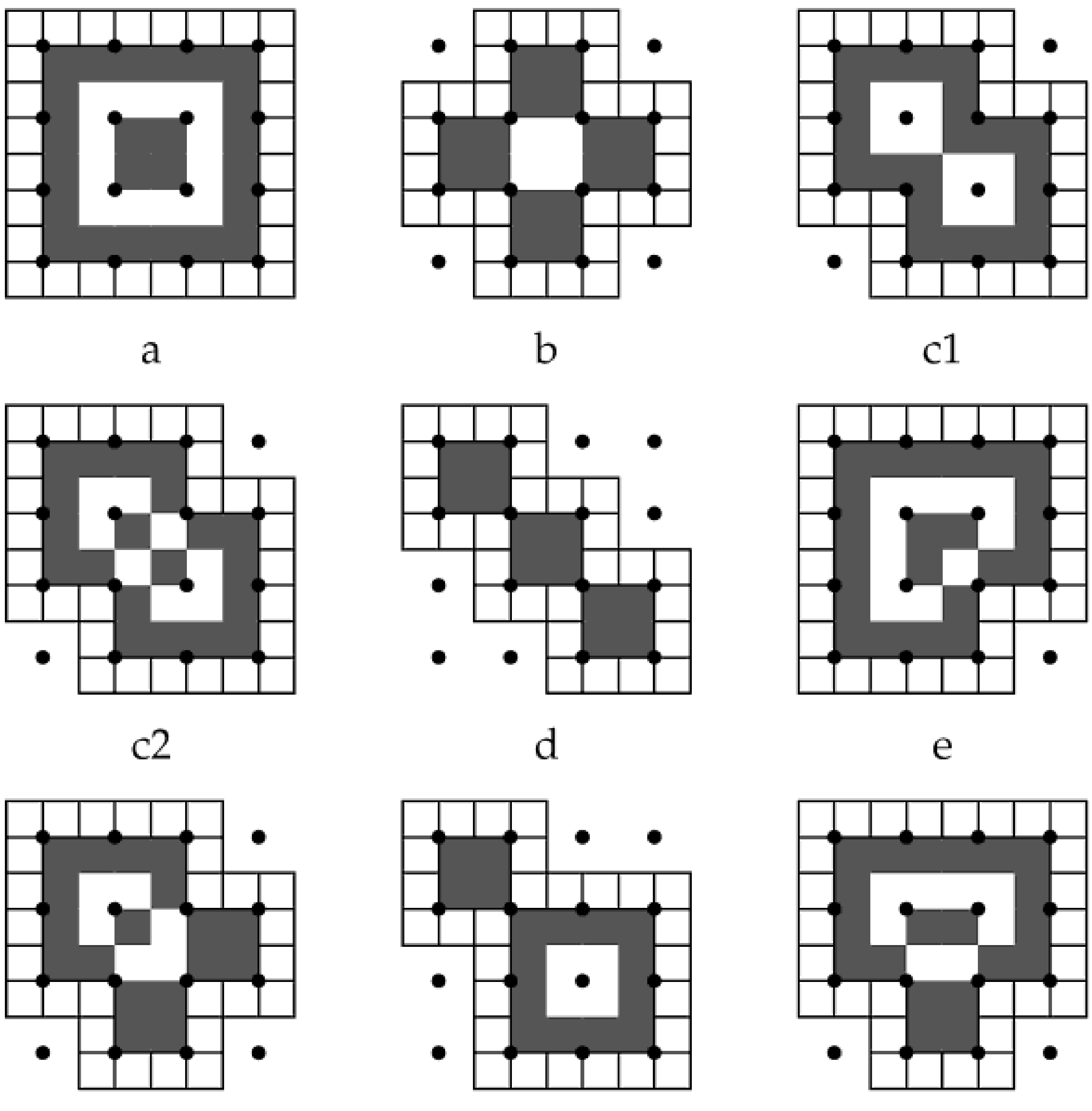


Figure 6.2

Figure 6.3 presents the twenty-seven symmetrical closed Lunda-36-ominoes with colored interiors. Figure 6.4 shows the seven possible symmetrical interiors of the second Lunda-36-omino in Figure 3.

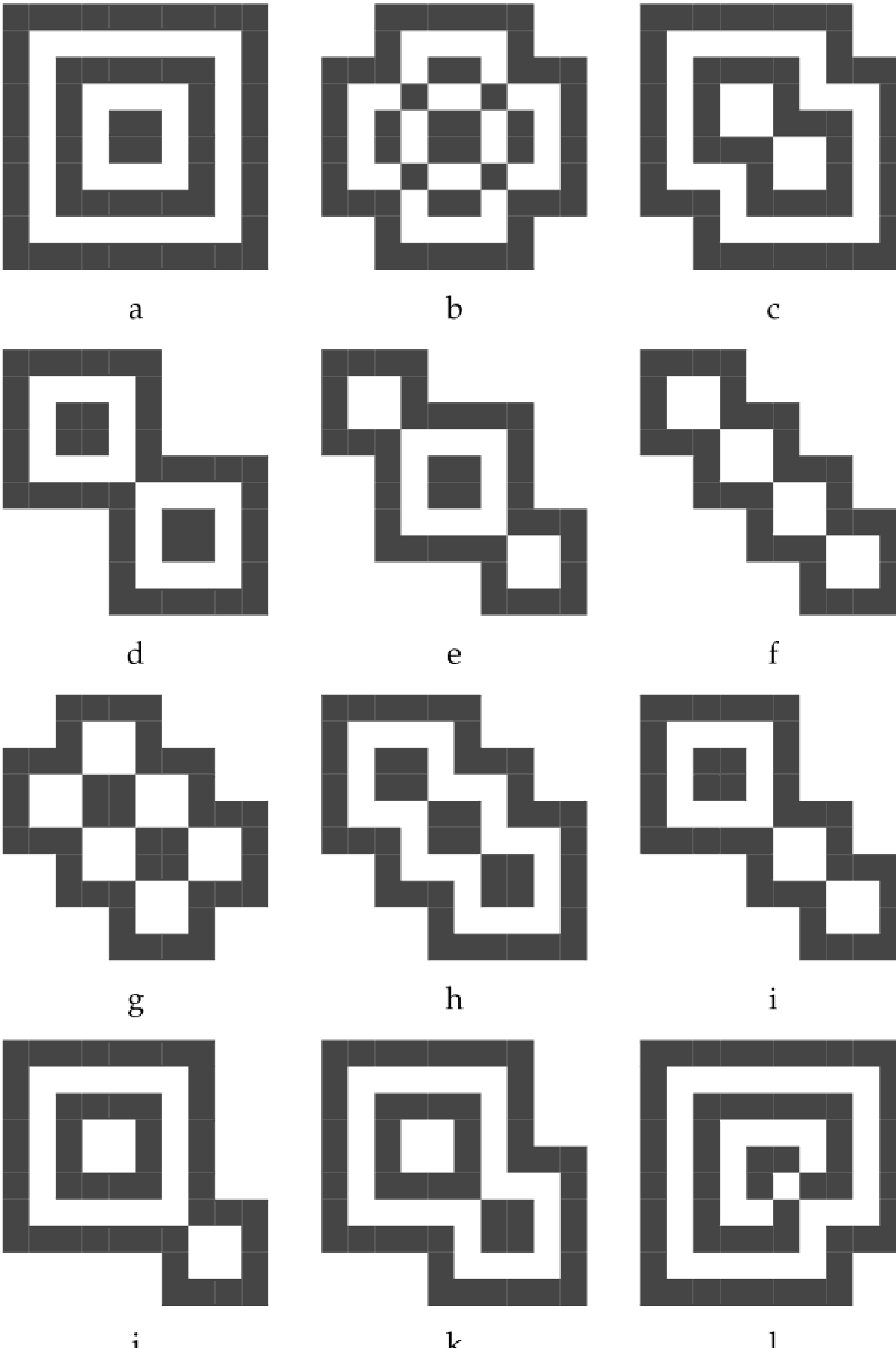


Figure 6.3 (1st part)

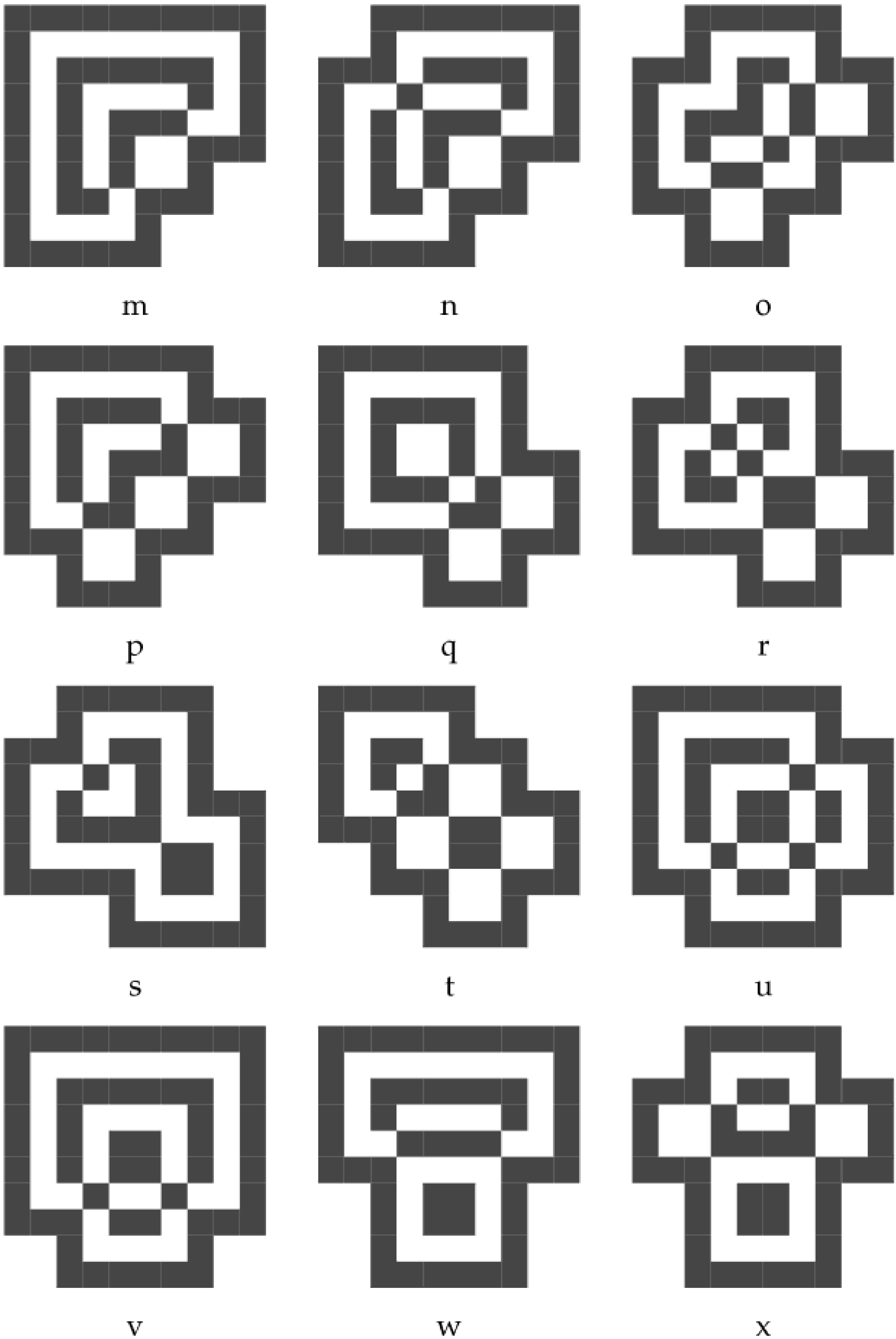
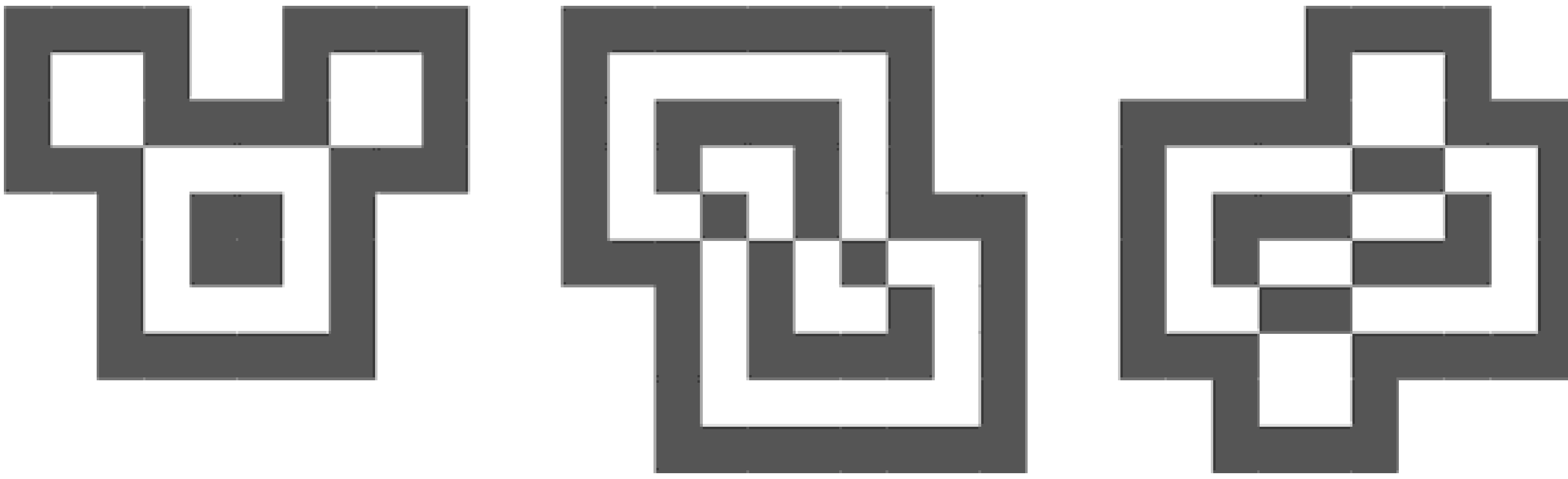


Figure 3 (2^{nd} part)



y z z1

Figure 3 (Conclusion)

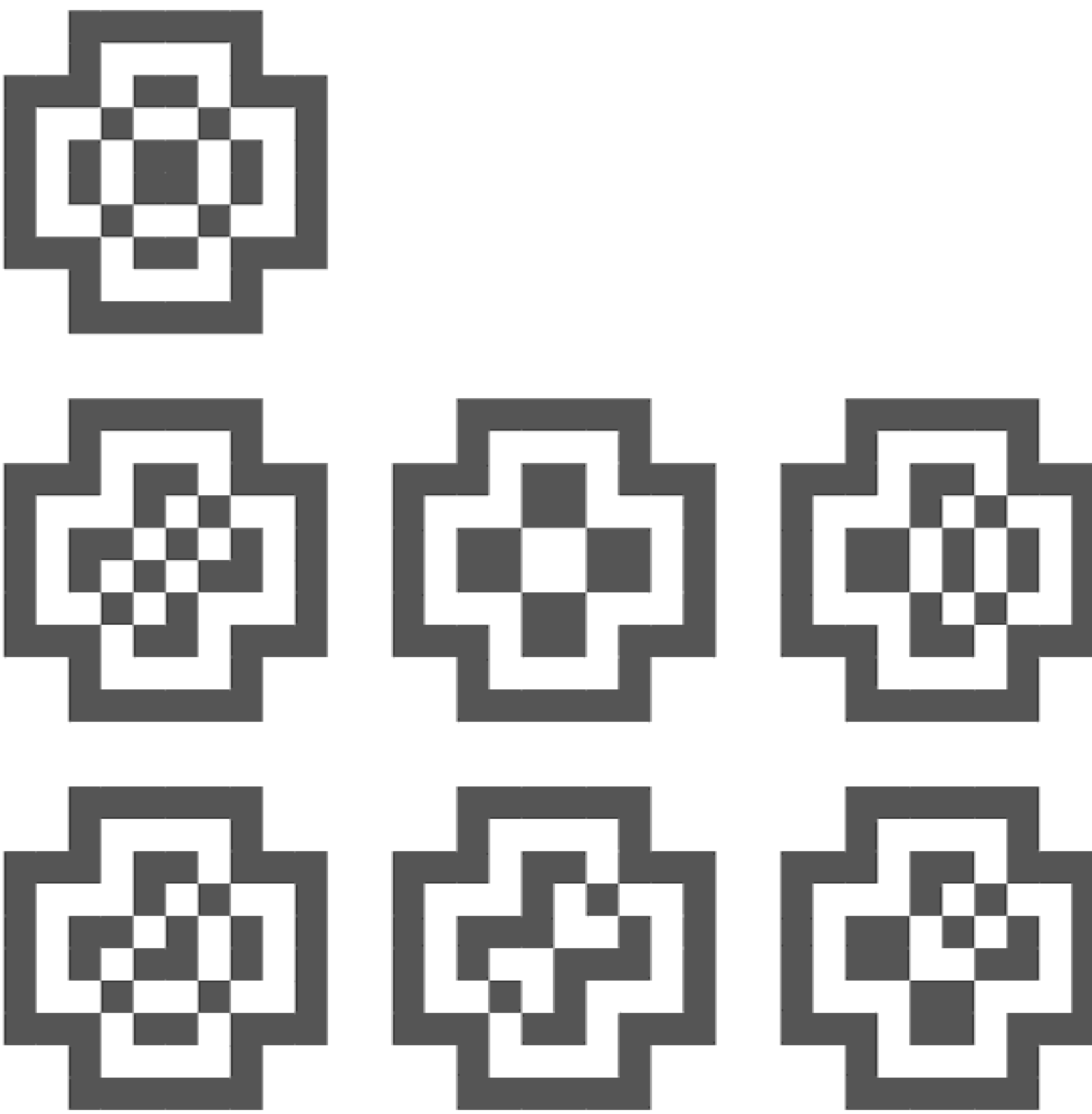


Figure 6.4

Figure 6.5 presents four closed Lunda-68-ominoes with their interiors colored. Each of these figures has four mirror symmetries, as do the Lunda-76-ominoes and Lunda-100-ominoes with colored interiors, in Figures 6.6 and 6.7 respectively.

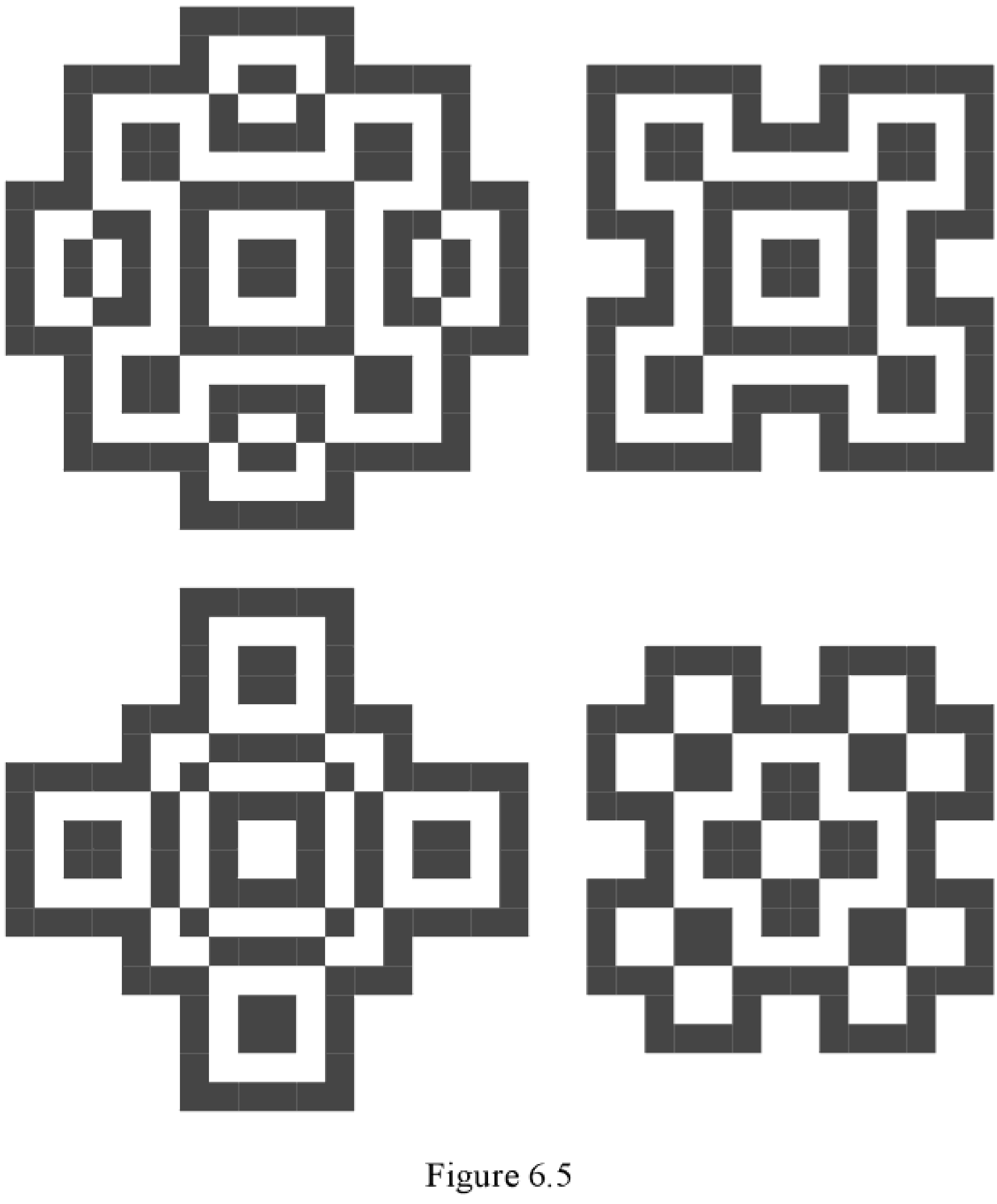


Figure 6.5

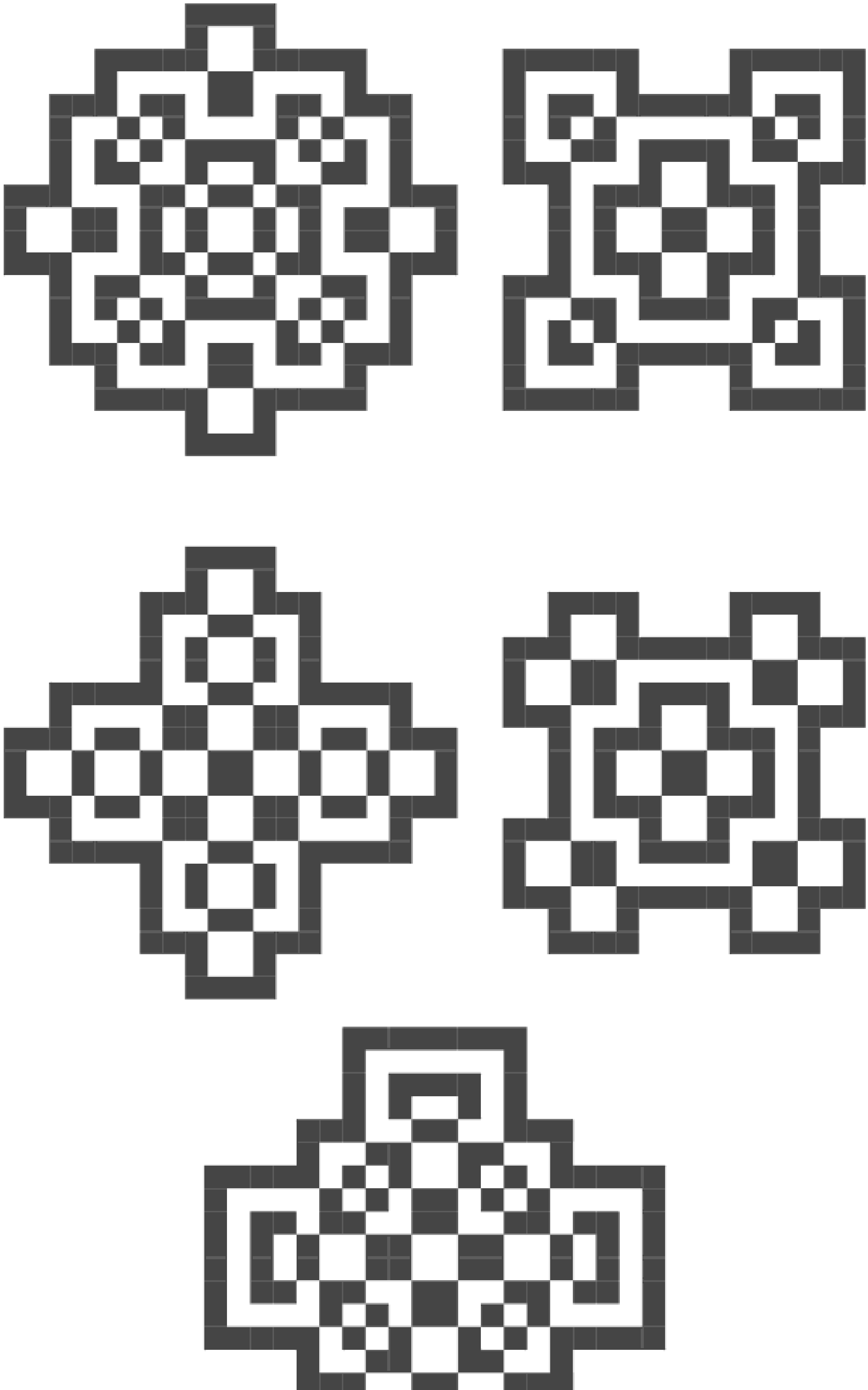


Figure 6.6

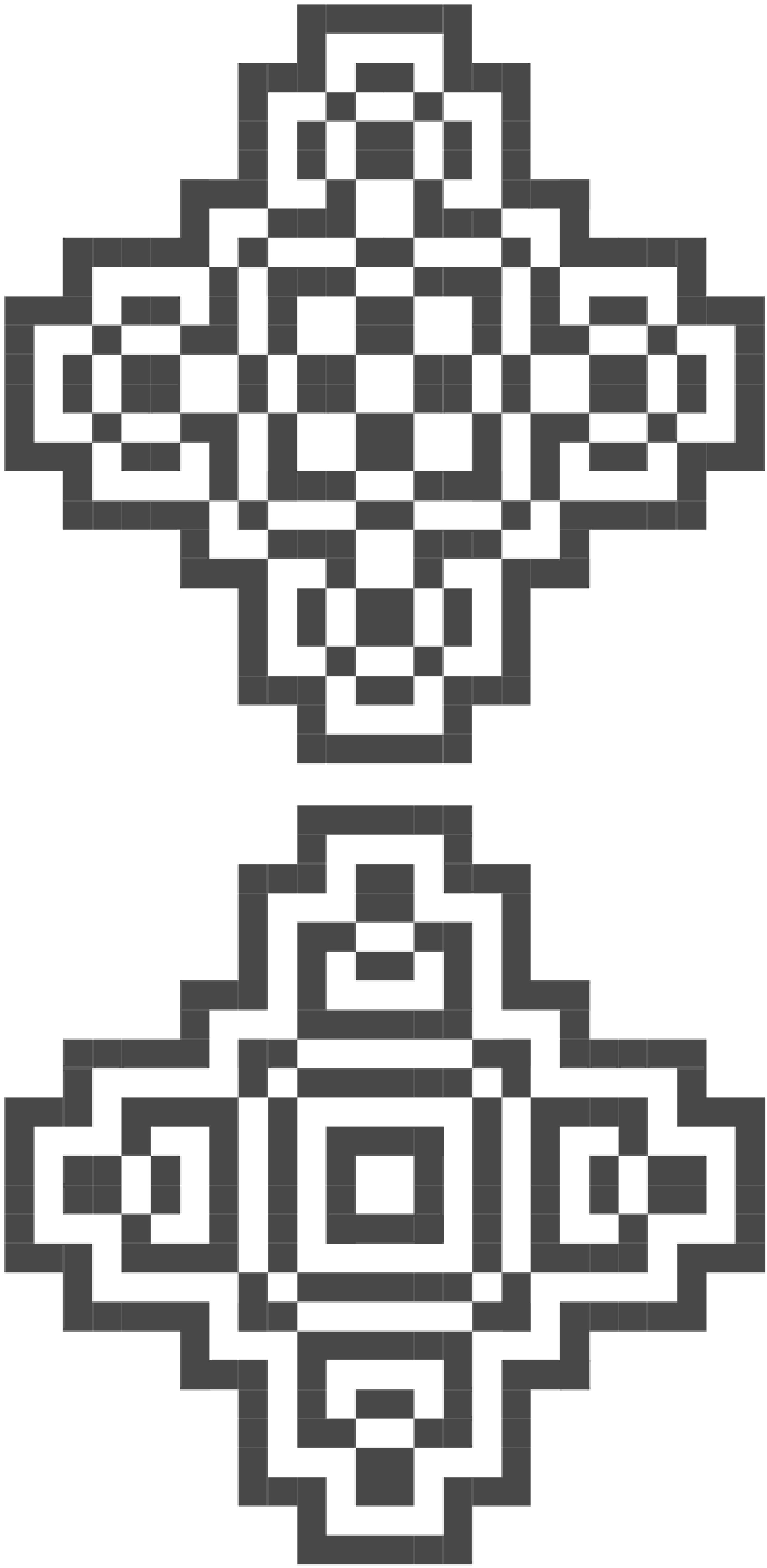
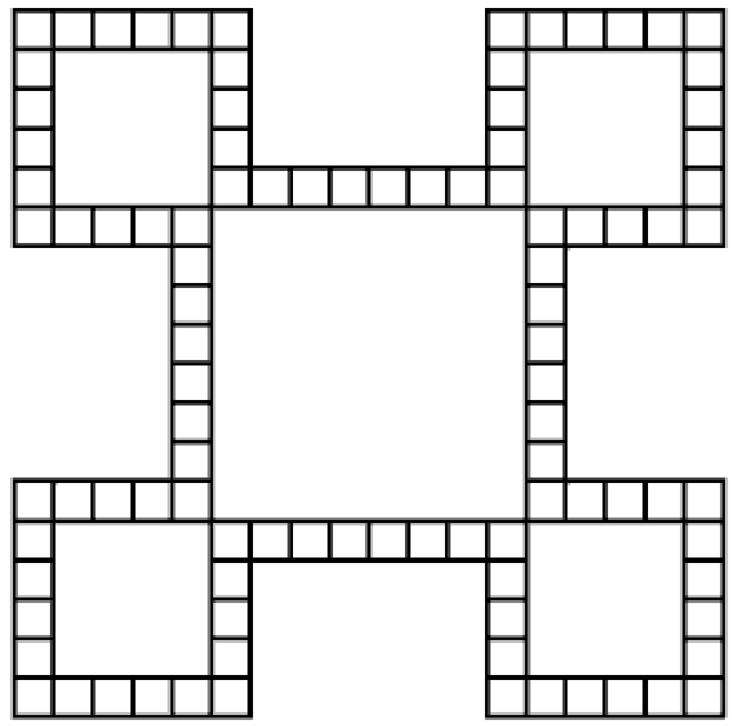
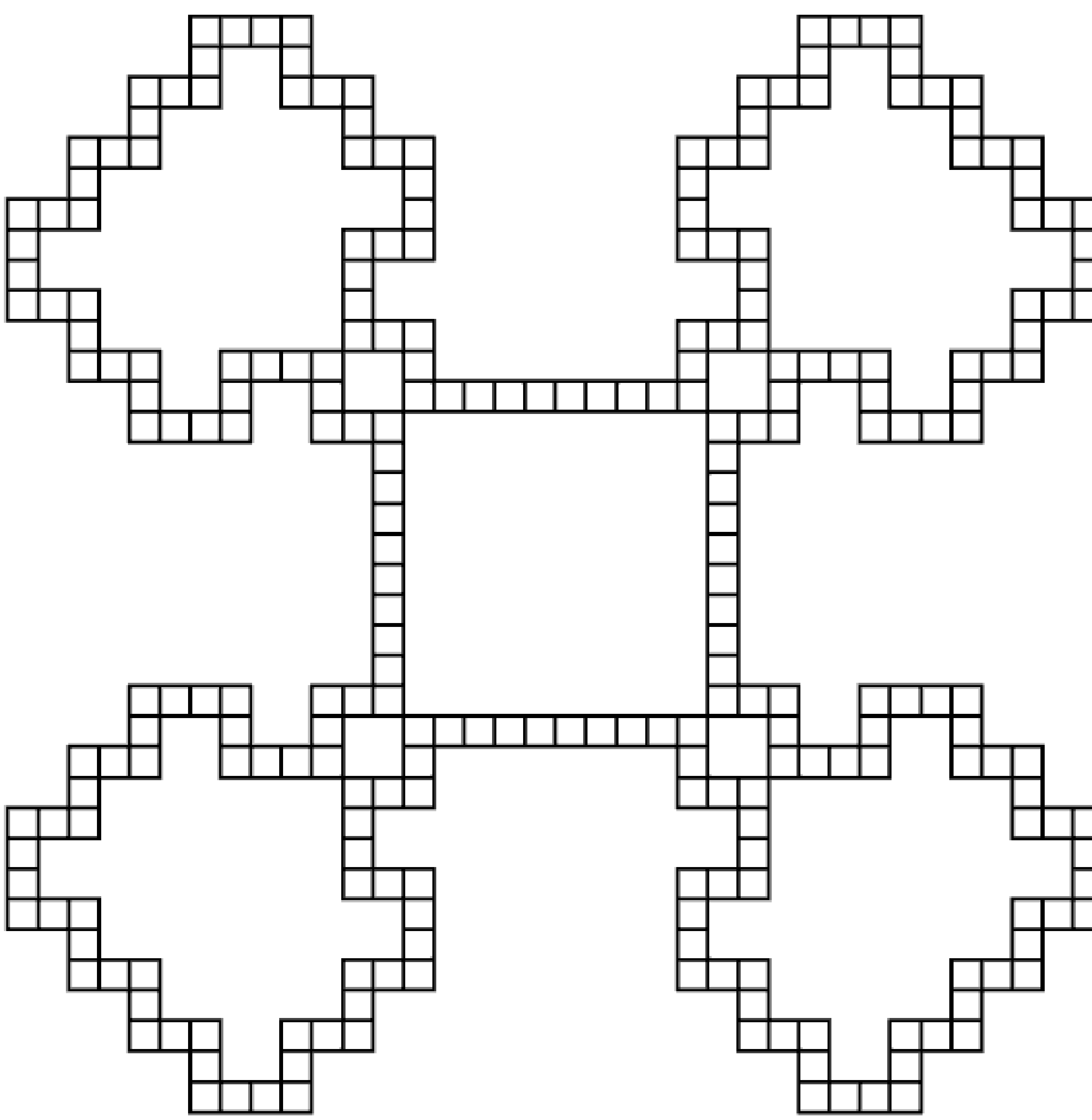


Figure 6.7

Figure 6.8 shows an example of a closed symmetrical Lunda-100-omino and Lunda-268-omino. Figure 6.9 shows them as part of symmetrical Lunda-designs of dimensions 28x28 and 40x40.

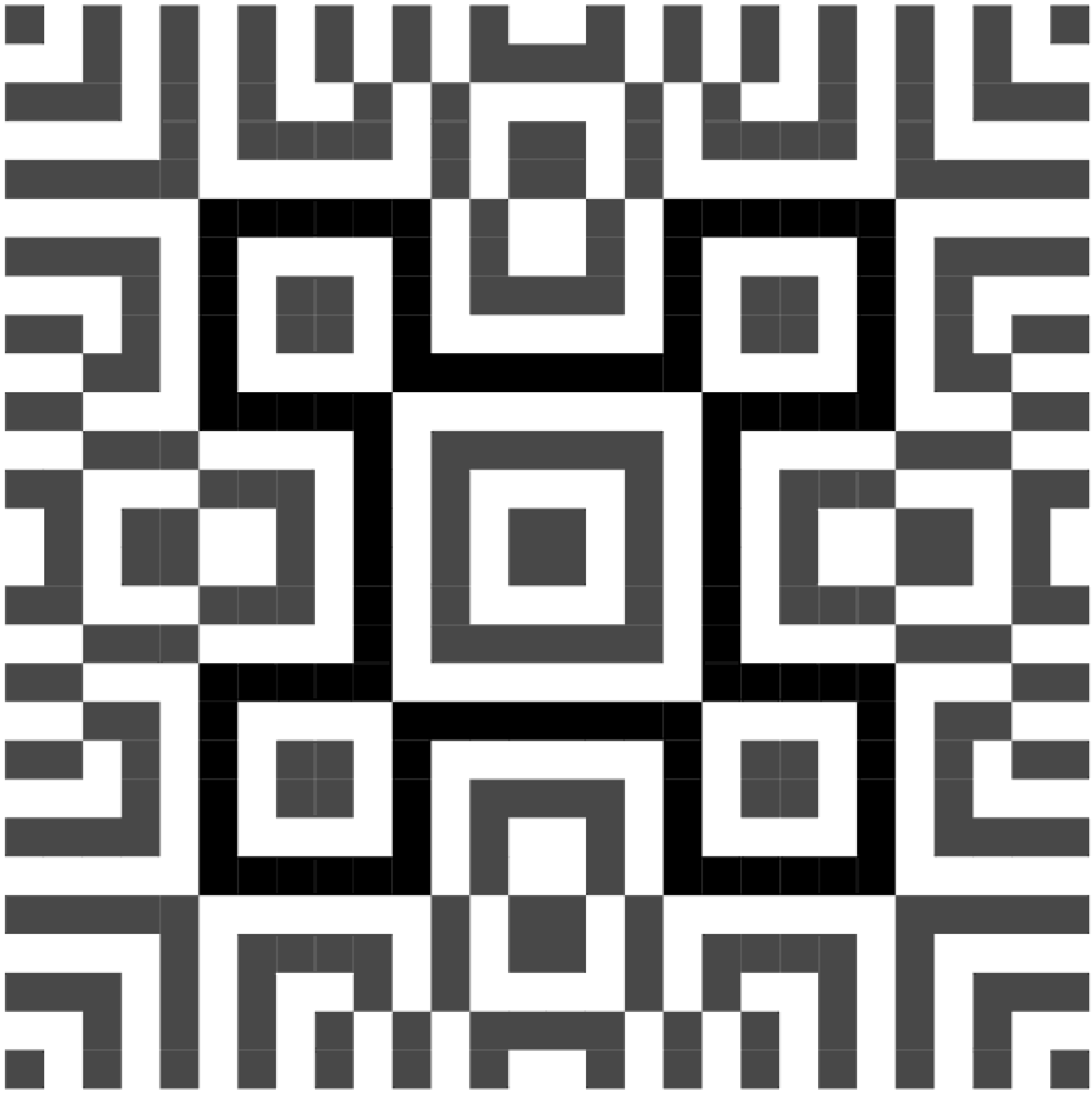


a

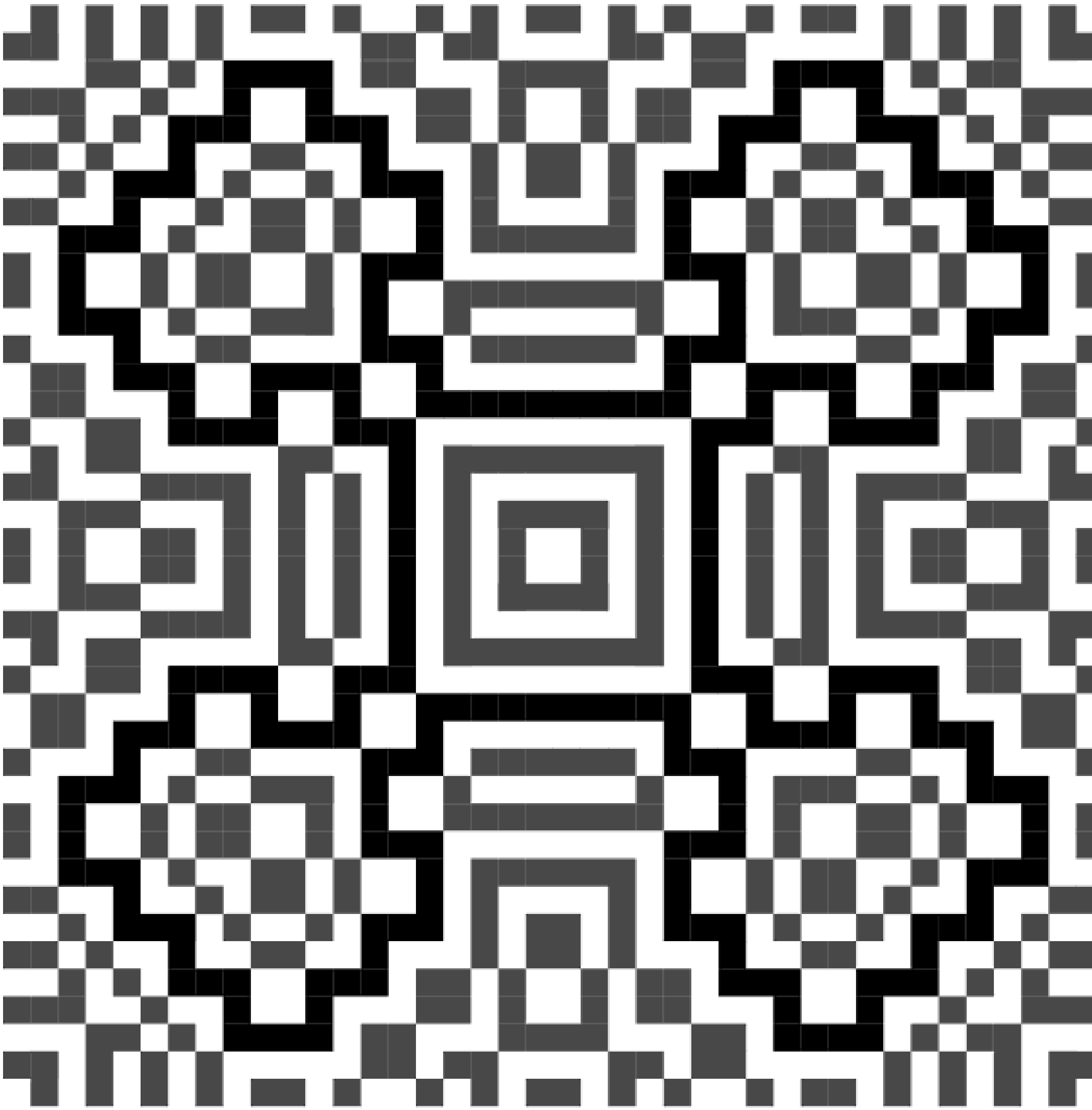


b

Figure 6.8



a
Figure 6.9



b

Figure 6.9

Chapter 7

EXAMPLES OF LUNDA-SPIRALS

Figure 7.1 shows part of two black (one shown as grey) and two white spirals, which together form an infinite Lunda-design with fourfold symmetry. The eight spirals in Figure 7.2 together constitute an infinite Lunda-design invariant under a half-turn. Figure 7.3 displays part of a zigzagging spiral embedded in an infinite Lunda-design.

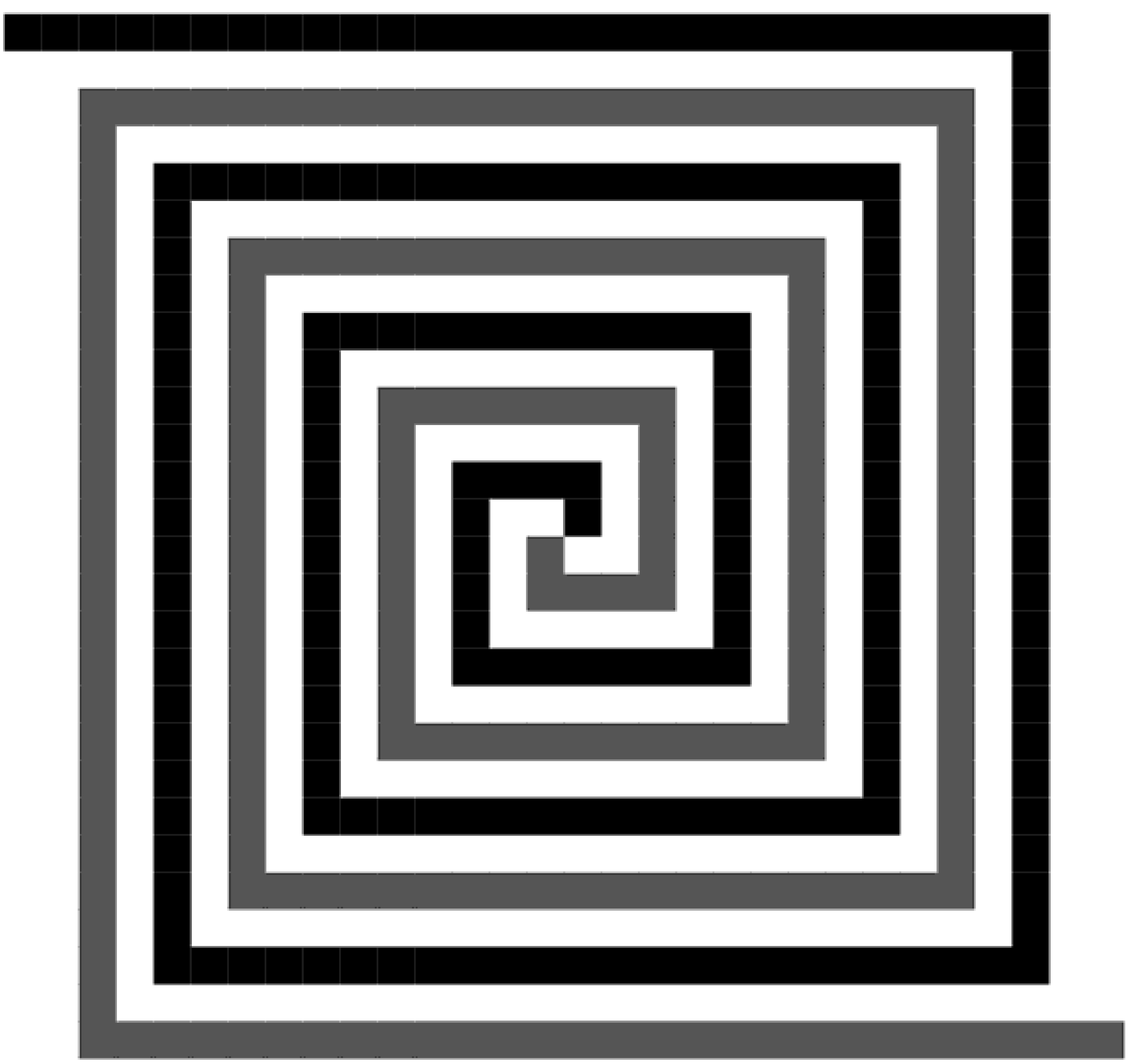


Figure 7.1

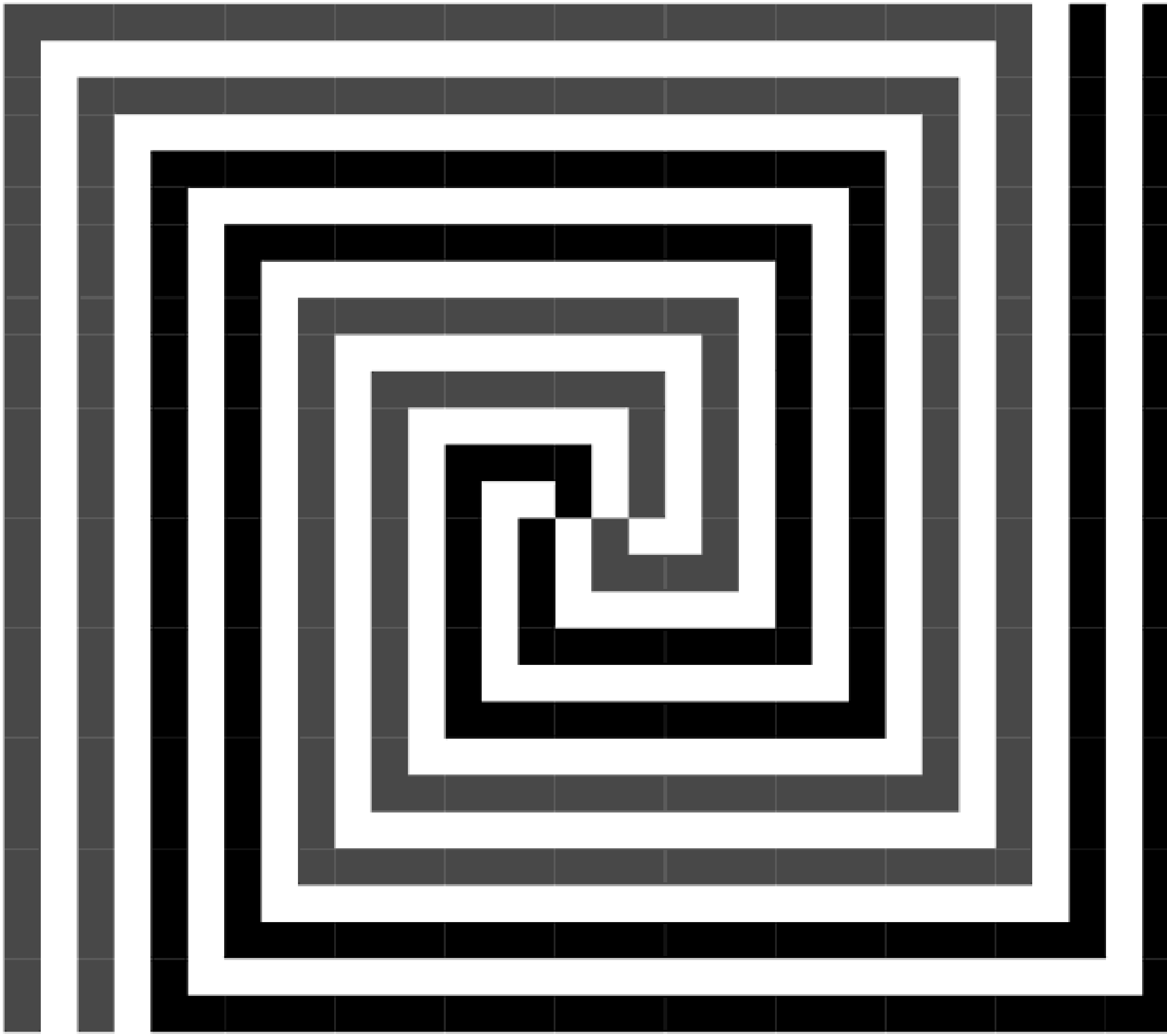


Figure 7.2

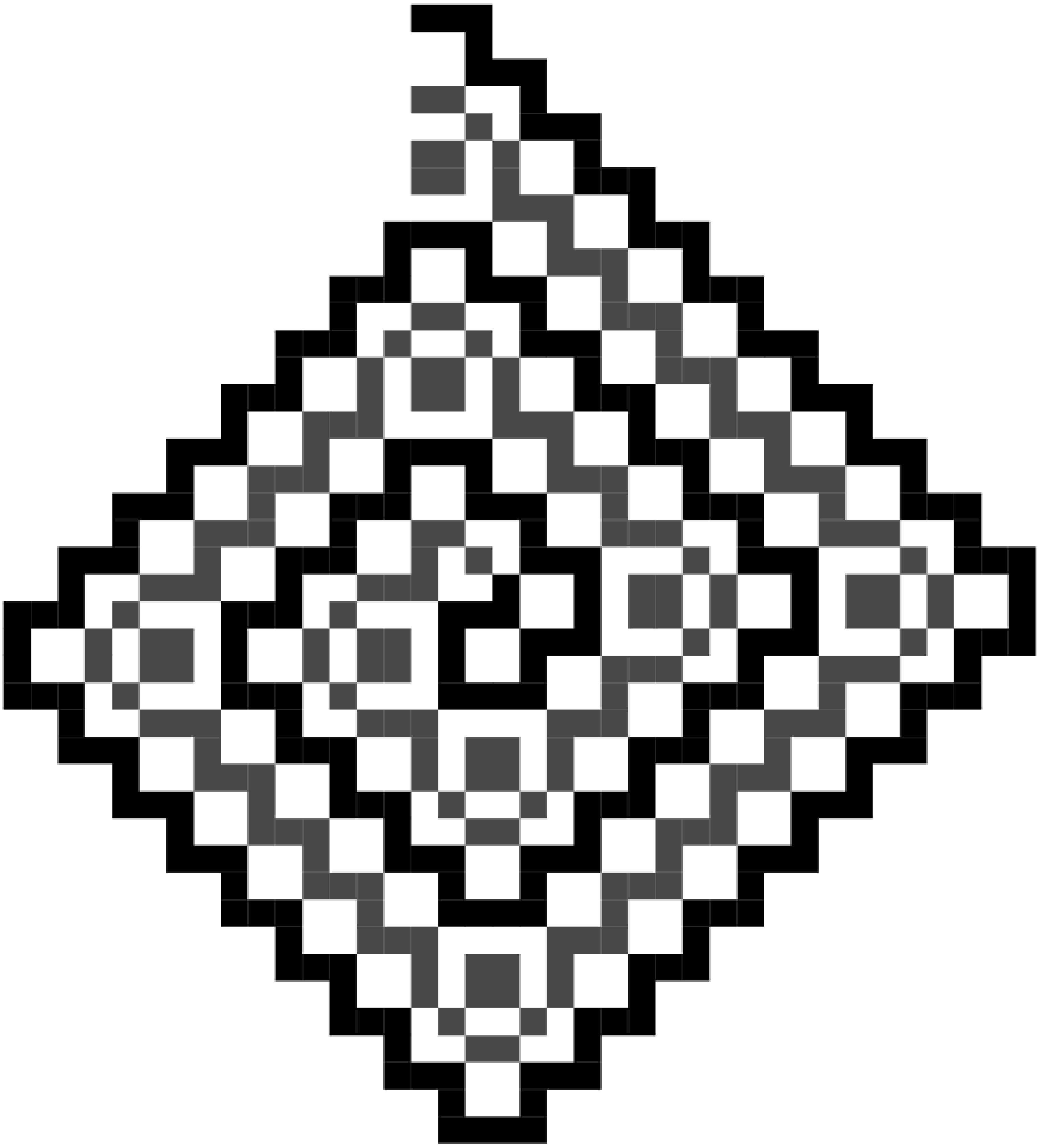


Figure 7.3

Chapter 8

LUNDA STRIP AND PLANE PATTERNS

The concept of Lunda-design may be extended in a natural way to one- and two-dimensional Lunda-patterns.

8.1 One-dimensional Lunda-patterns (Lunda-strip-patterns)

Consider an infinite grid, IG(n), having as points (2p,2q-1) with $0<q\leq n$, where p denotes a whole number, and q and n natural numbers. Figure 8.1 displays IG(2).

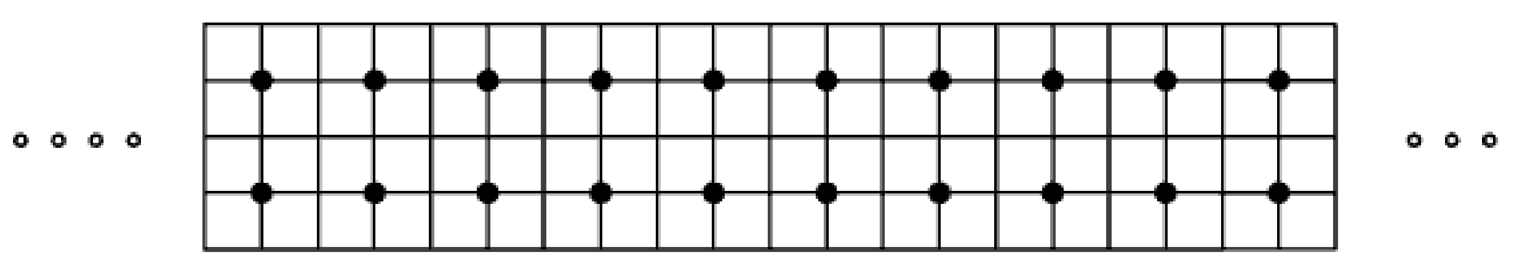
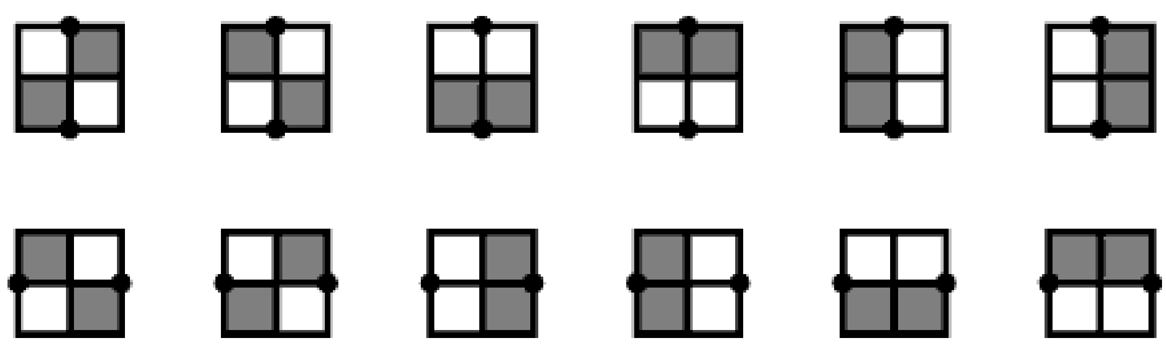


Figure 8.1

A **one-dimensional Lunda-pattern** (of height n) may be defined as a black-and-white pattern (between the horizontal lines y=0 and y=2n) with the following characteristics:

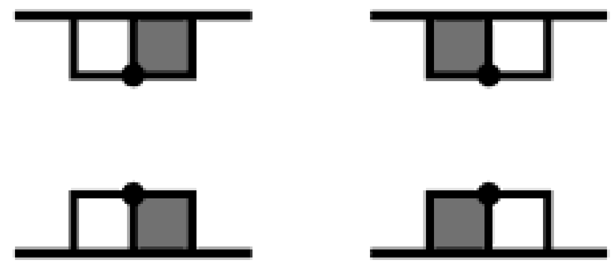
(i) Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are always black and two are white (see Figure 8.2a);



a

Figure 8.2

(ii) Of the two border unit squares of any grid point in the first or last row, one is always black and the other white (see Figure 8.2b).



b

Figure 8.2

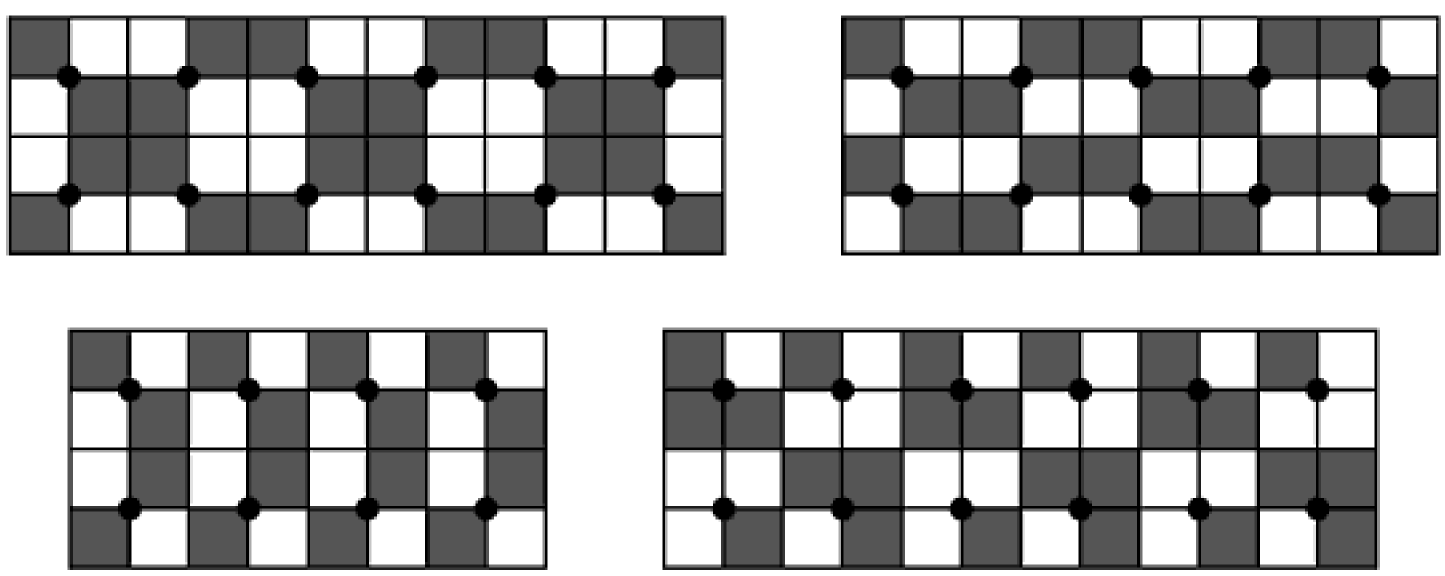


Figure 8.3

Pattern means that the design has translation symmetry. Figure 8.3 presents examples of one-dimensional Lunda-patterns of height 2. In each of the examples, we also have — as in the case of finite Lunda-designs — a third characteristic:

(iii) In each column there are as many black as white unit squares.

If this happens, we call the one-dimensional Lunda-pattern **strong**. Figure 8.4 shows a Lunda-strip-pattern of height 2 that is not strong.

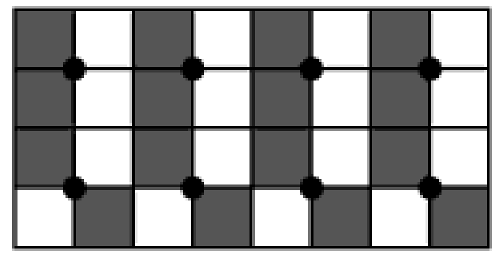


Figure 8.4

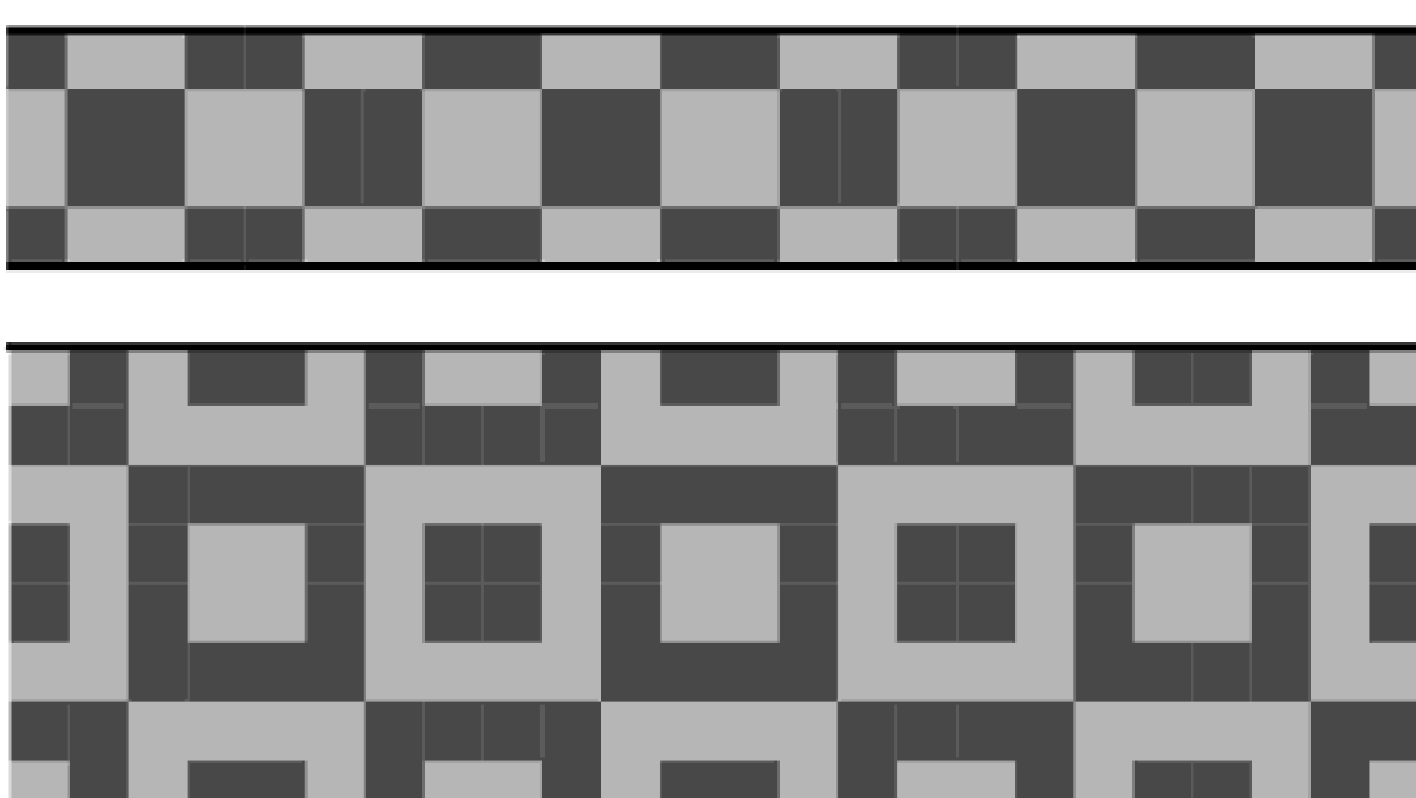
Lunda-strip-patterns may be either one-color or two-color patterns. A pattern is called a two-color pattern if there is some rigid motion (rotation, translation, mirror reflection, or glide reflection) of the strip, which interchanges the colors everywhere. Classification by symmetry results in seven one-color and seventeen two-color strip pattern classes. For each of the twenty-four one-color and two-color classes it is possible to construct Lunda-strip-patterns which belong to it, as the examples in Figure 8.5 show (with the grid points unmarked). We use the internationally accepted notation of Belov (1956), whereby each class is indicated by four symbols *pxyz* as follows:

(p) The first symbol is *p* if no translation reverses the two colors; it is *p' if* some translation reverses the colors.

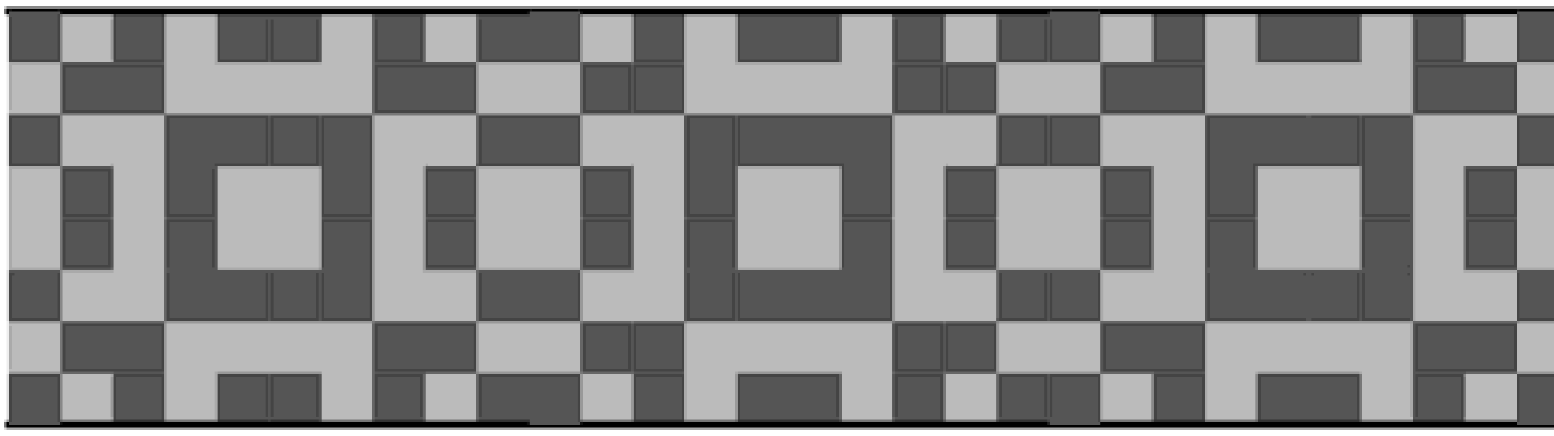
(x) The second symbol, *x*, is *1* if there is no vertical reflection consistent with color; *m* if there is a vertical reflection which preserves color; *m'* if all vertical reflections reverse the colors;

(y) The third symbol, *y*, is *1* if there is no horizontal reflection or glide reflection; *m* if there is a horizontal reflection which preserves color; *m' if* there is a horizontal reflection which reverses colors (except in the two cases beginning with *p',* in which two cases *y* is *a*); *a' if* there is no horizontal reflection, but the shortest glide reflection reverses colors; and is *a* otherwise.

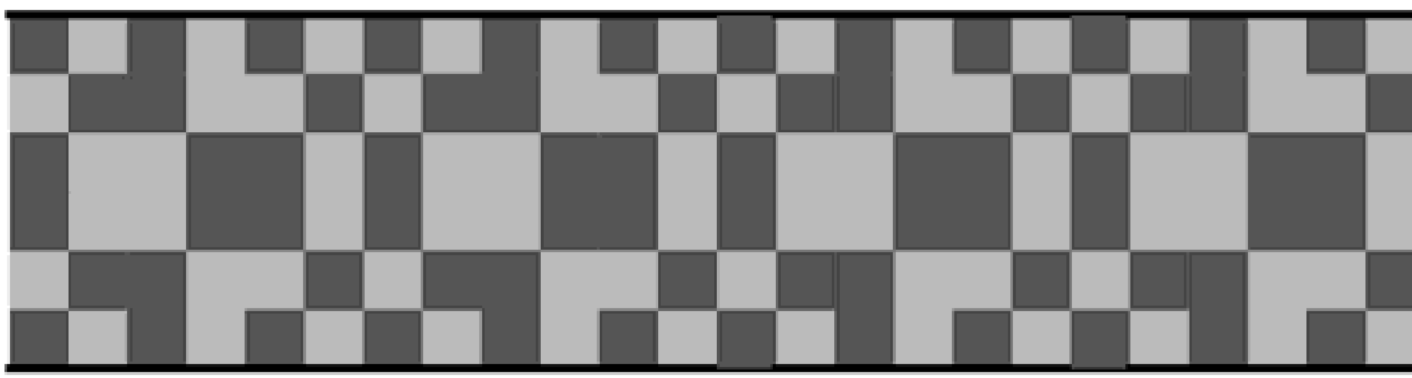
(z) The fourth symbol, *z*, is *1* if there no half-turn consistent with color; *2* if there are half-turns which preserve color; *2'* if all half-turns reverse colors (cf. Washburn & Crowe, 1988, 69).



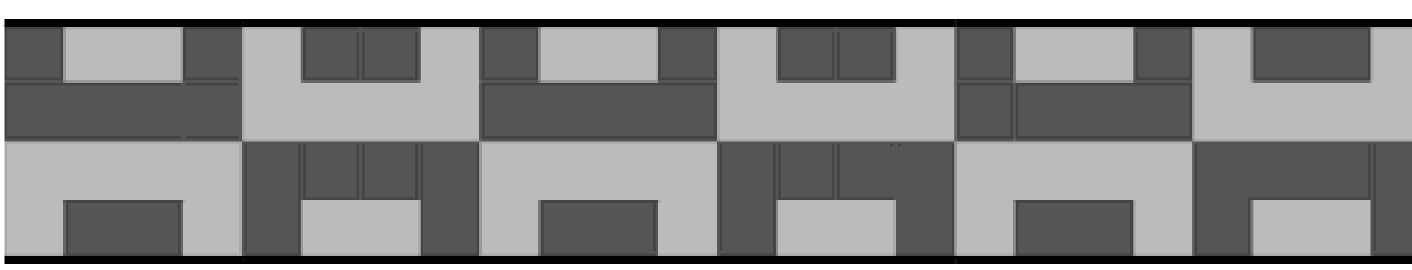
a: *p'mm2*
Figure 8.5 (1st part)



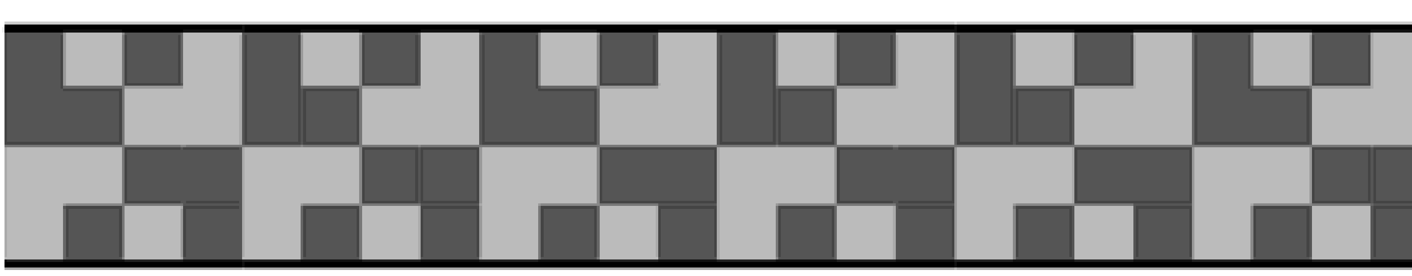
b: *pmm2*



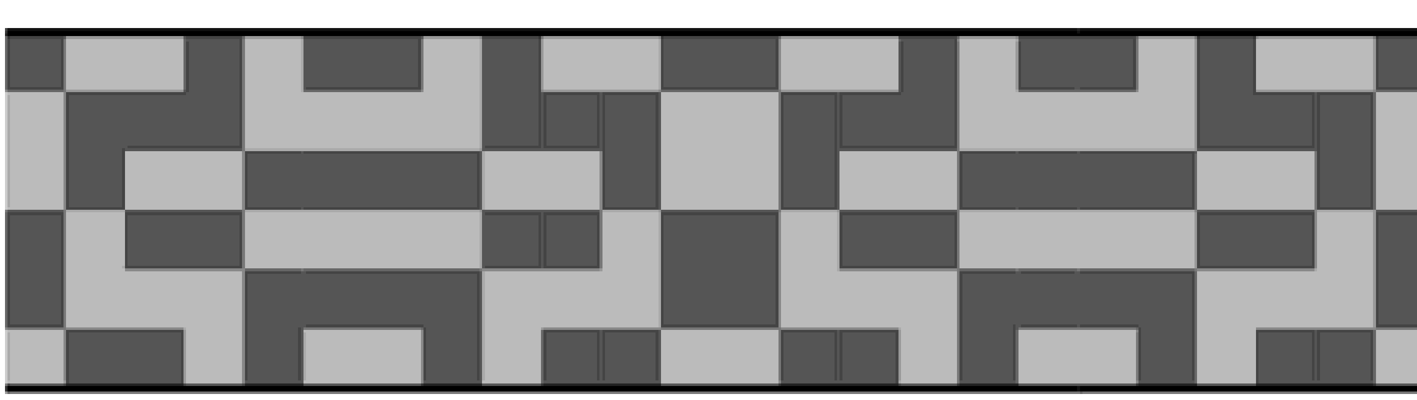
c: *pm'm2'*



d: *p'ma2*

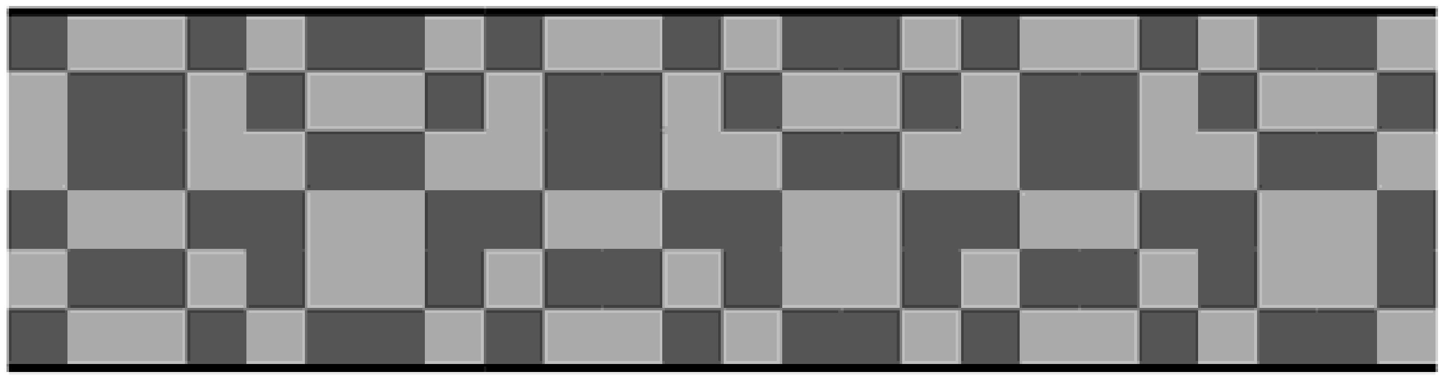


e: *pm'm'2*

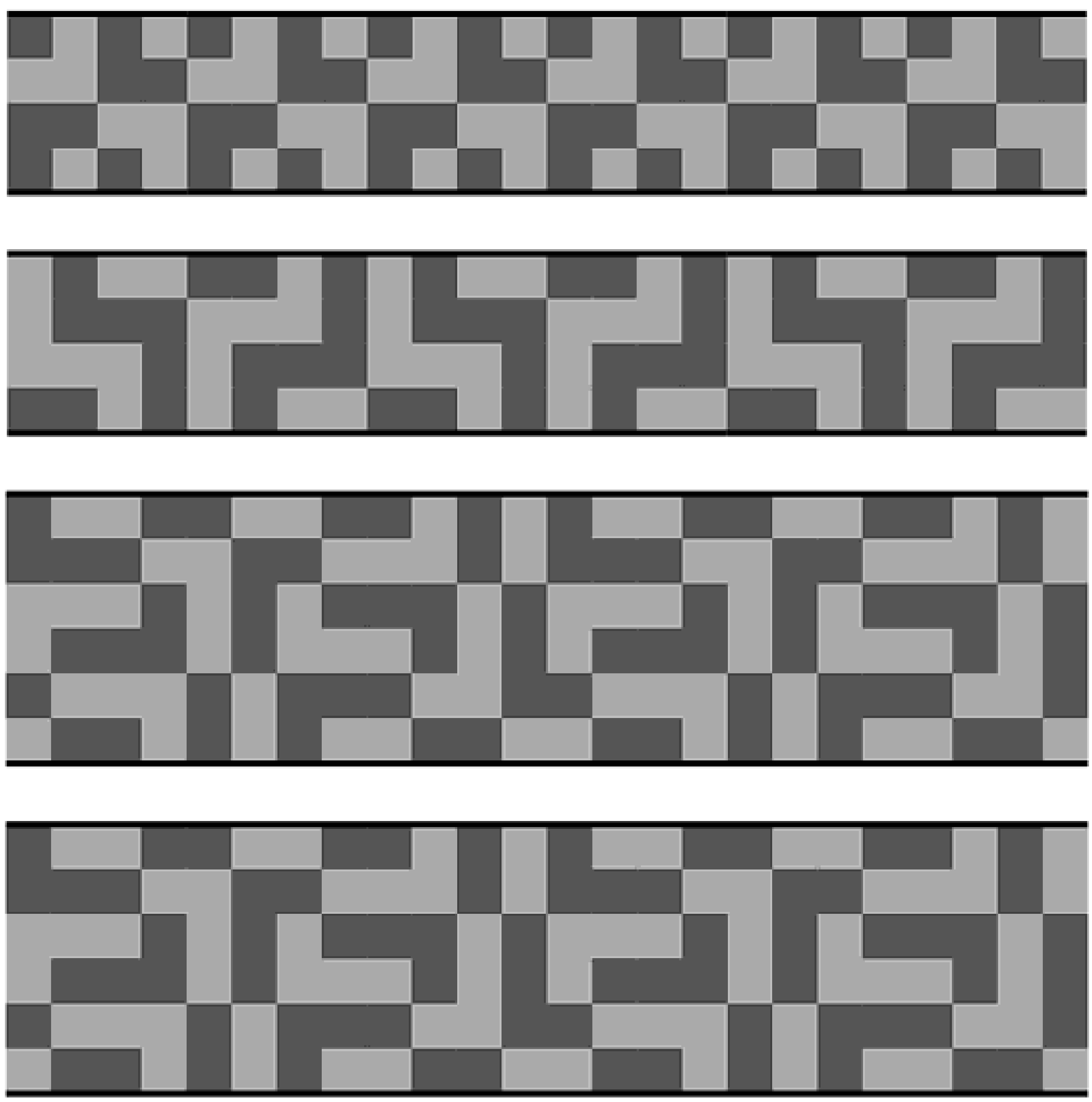


f: *pmm'2'*

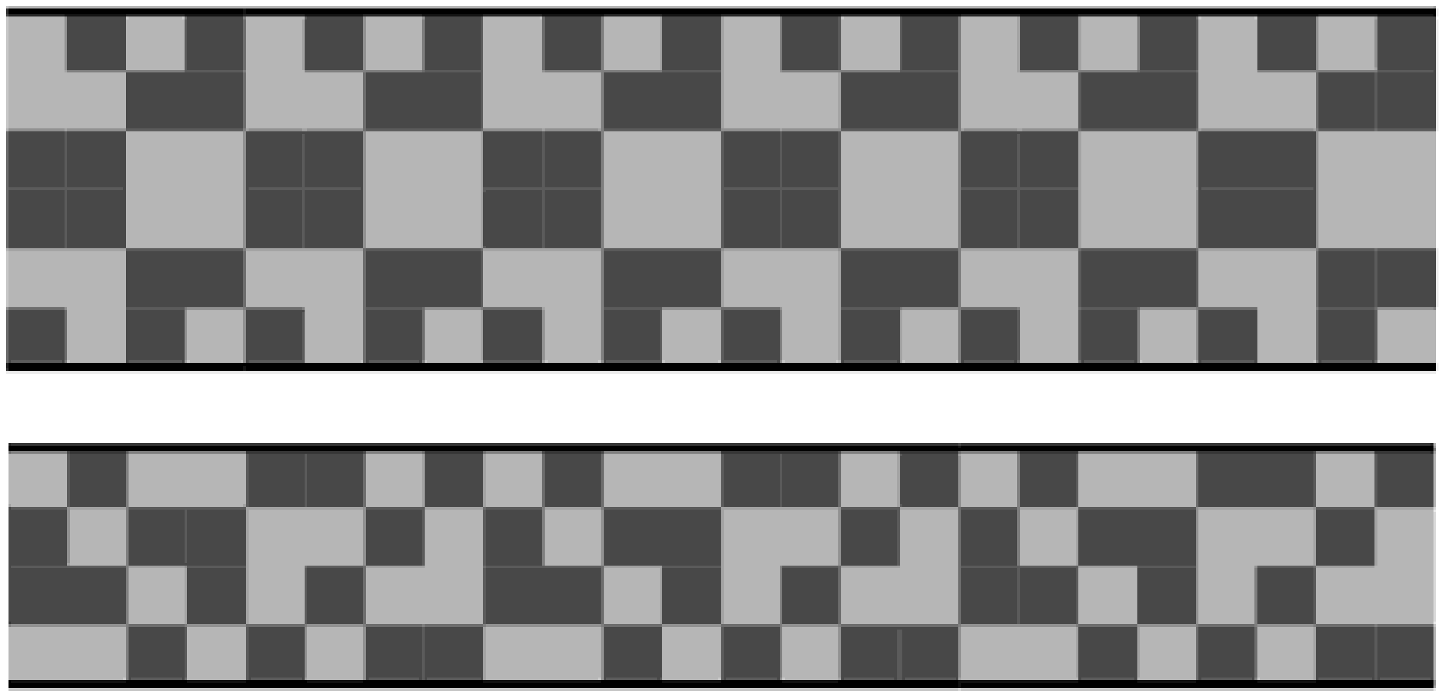
Figure 8.5 (continued)



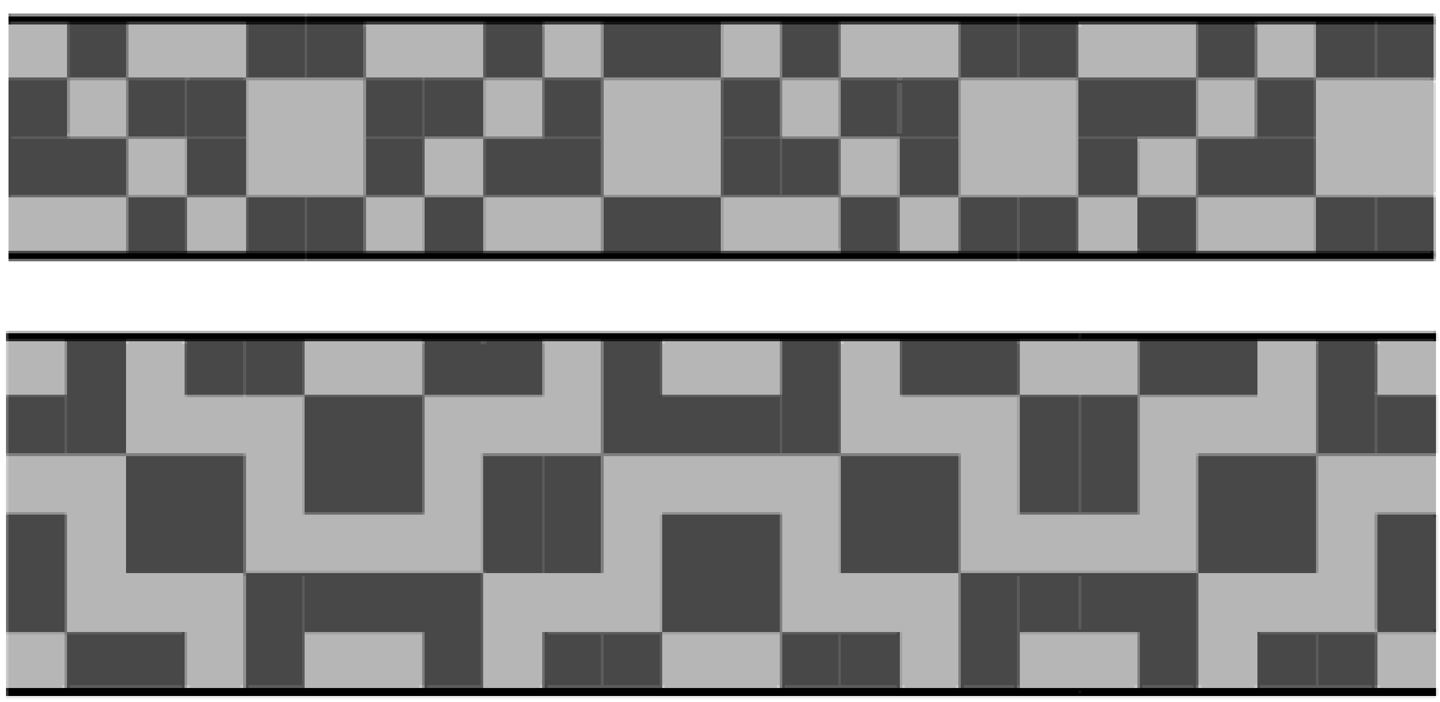
g: *pma'2'*



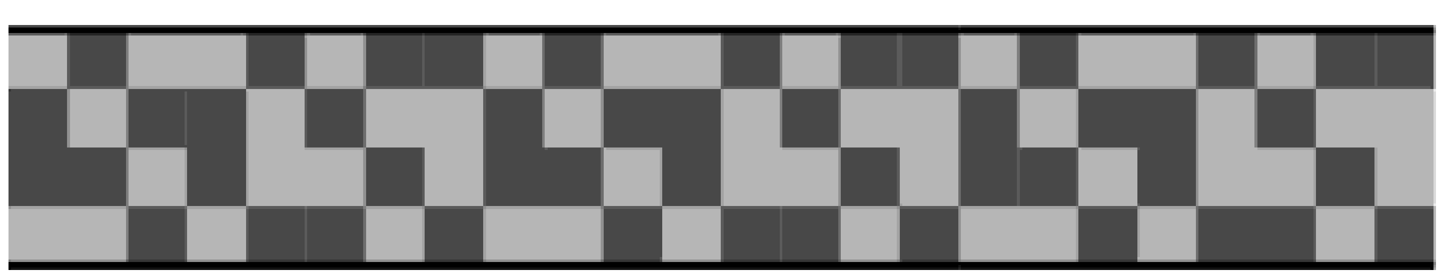
h: *pm'a2'*
Figure 8.5 (continued)



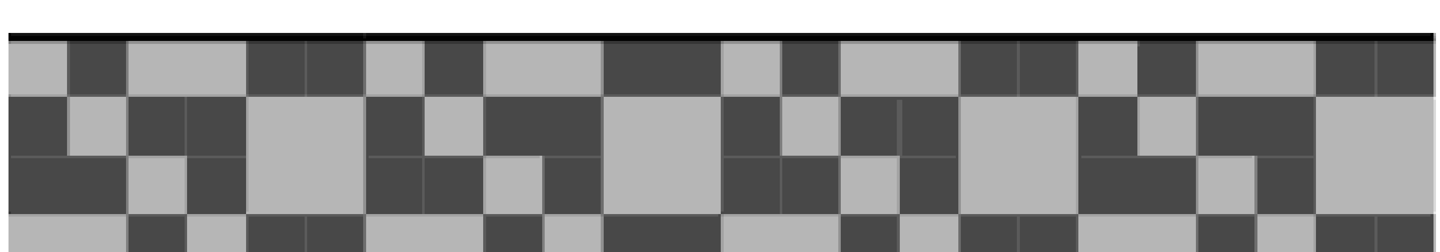
i: *pm'a'2*



j: *pma2*

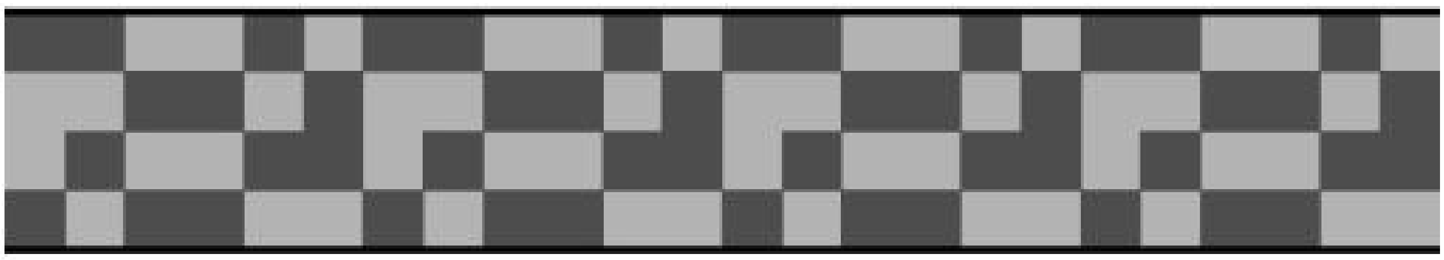


k: *p'112*

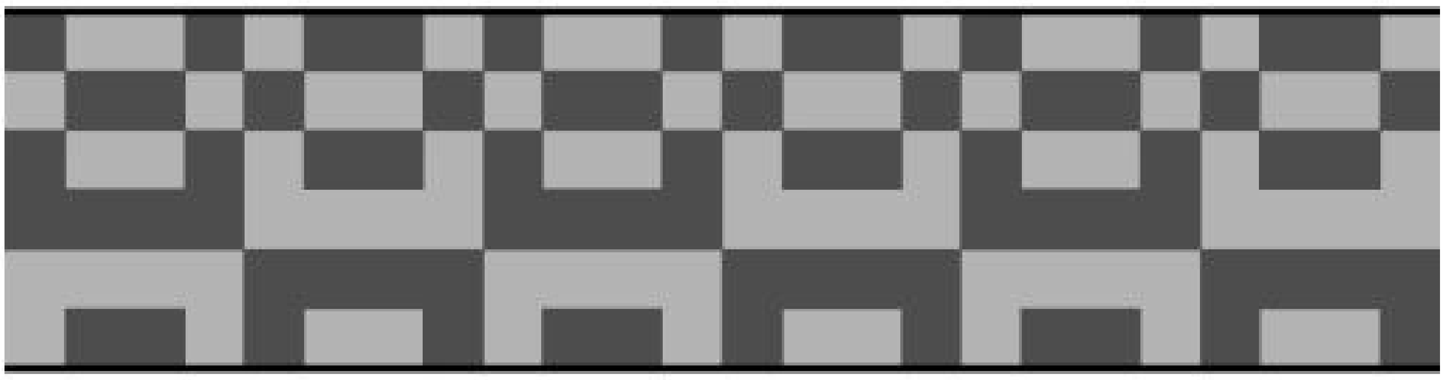


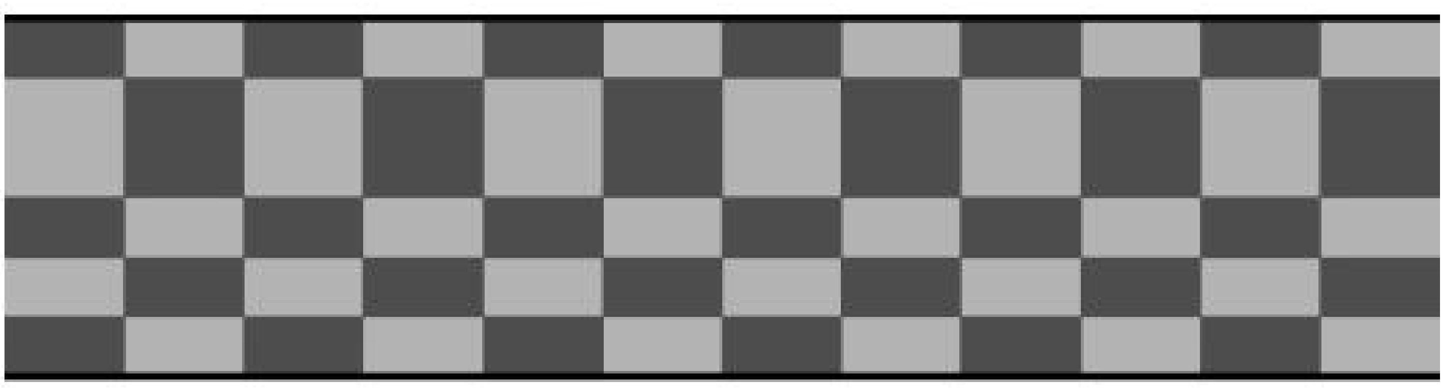
l: *p112*

Figure 8.5 (continued)

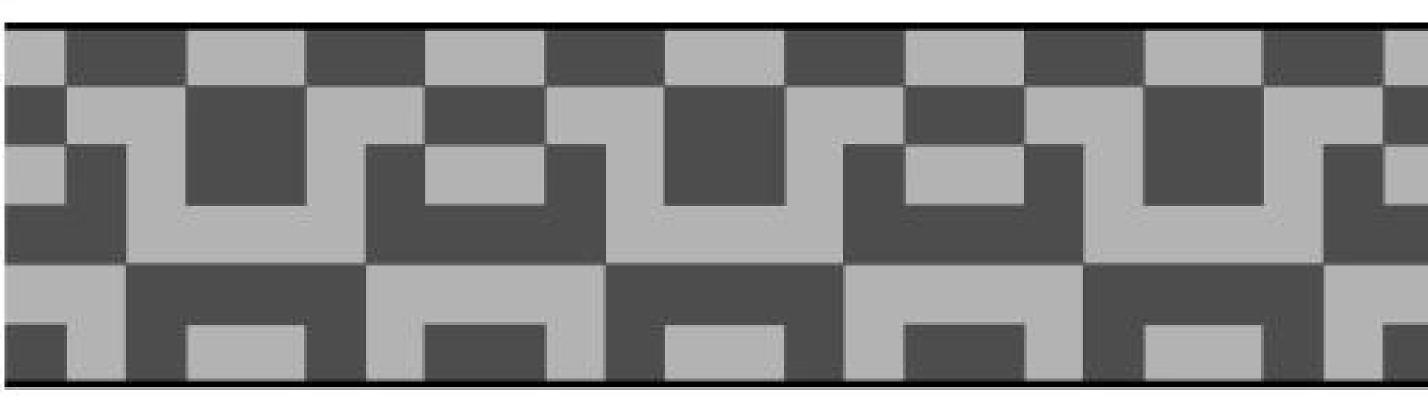


m: *p112'*

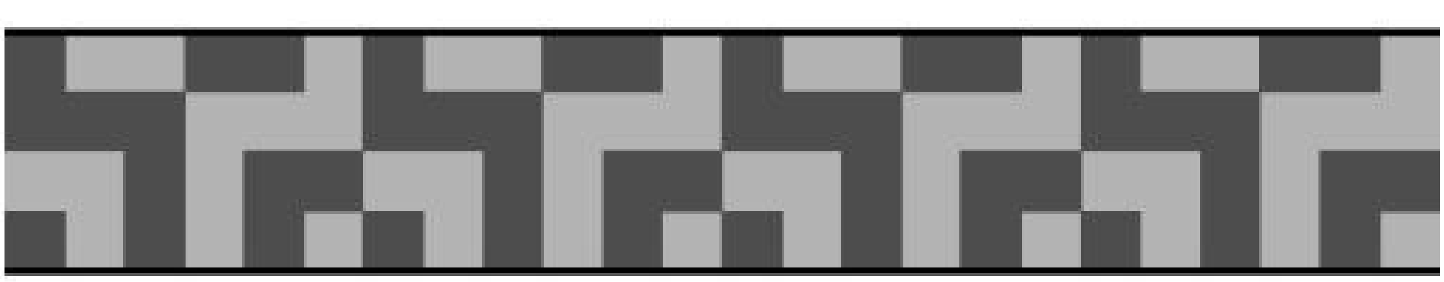


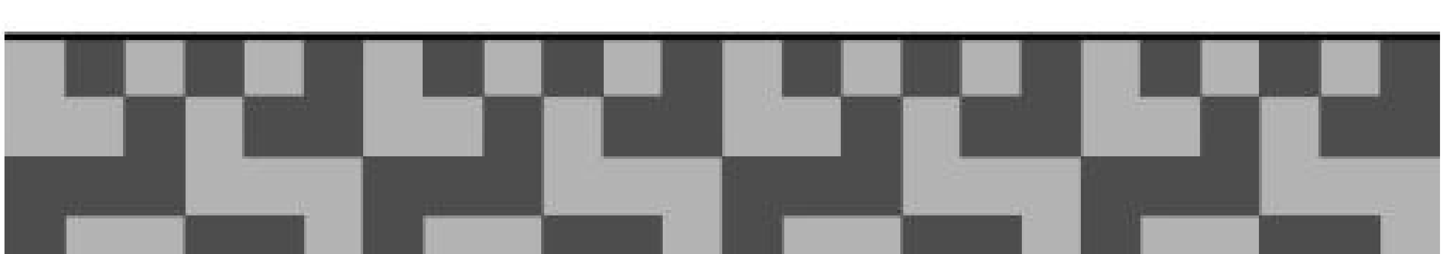


n: *p'm11*



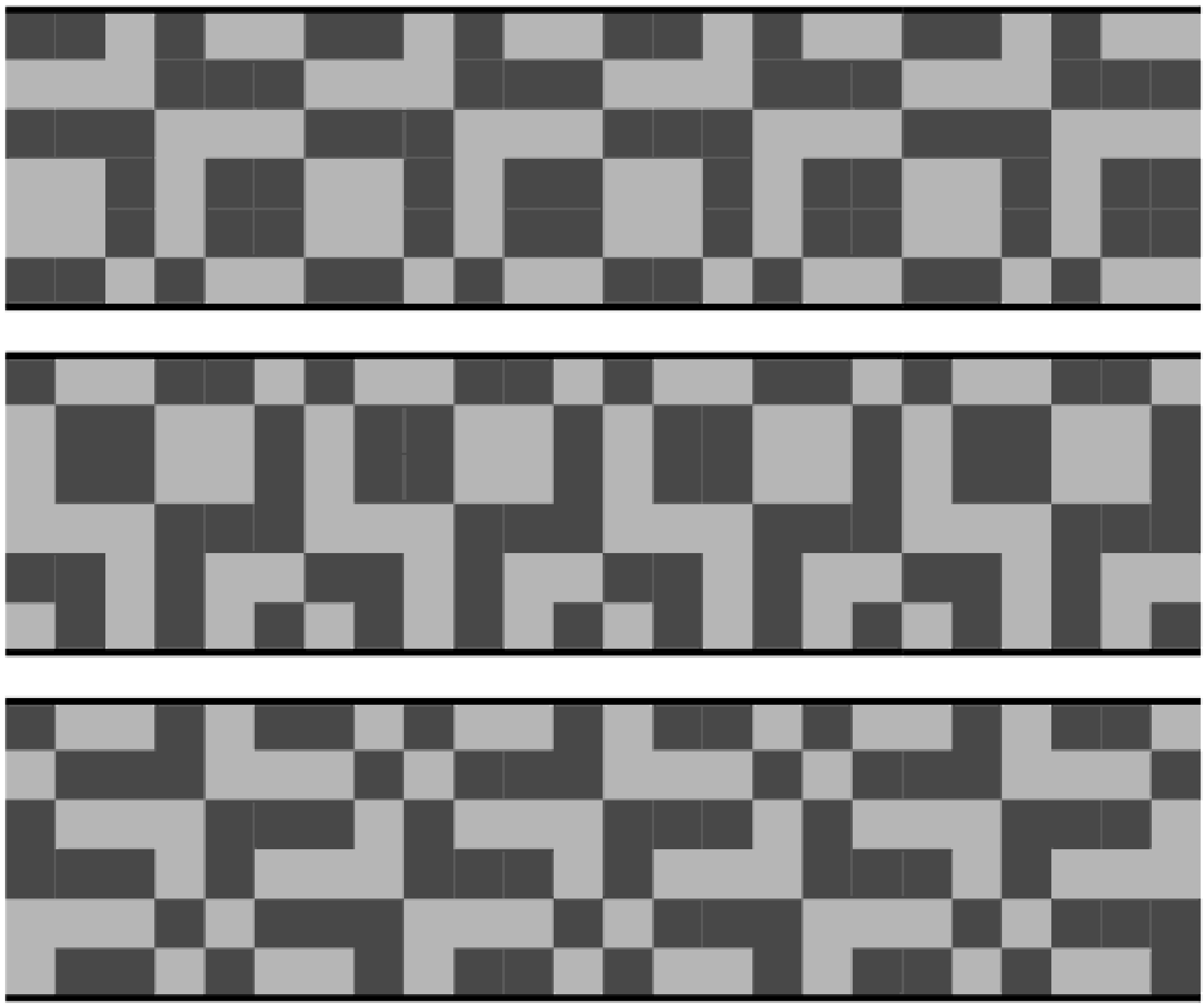
o: *pm11*



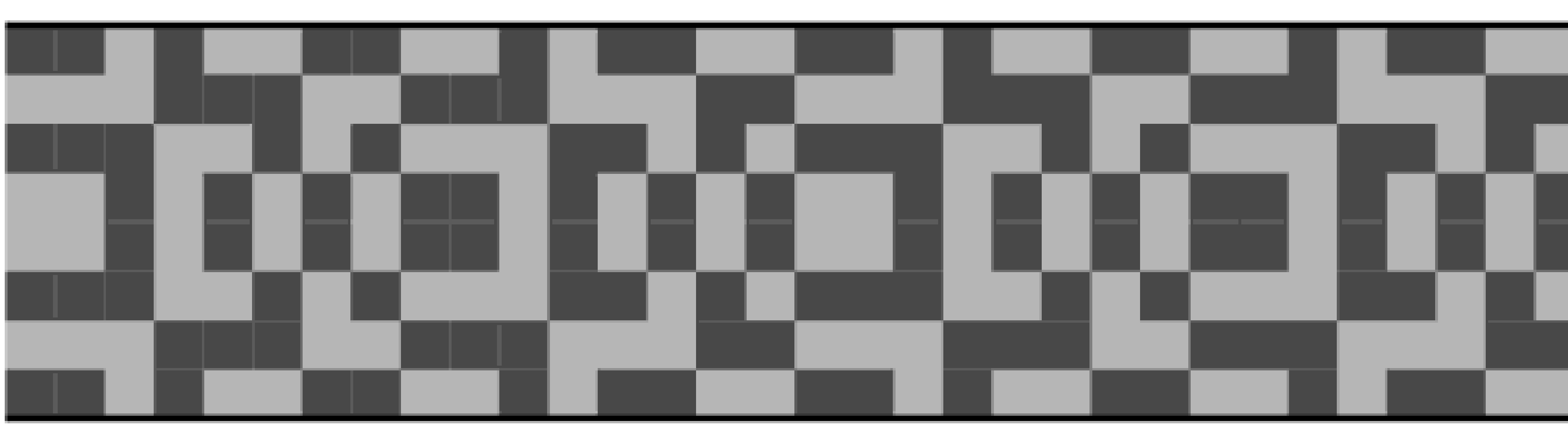


p: *pm'11*

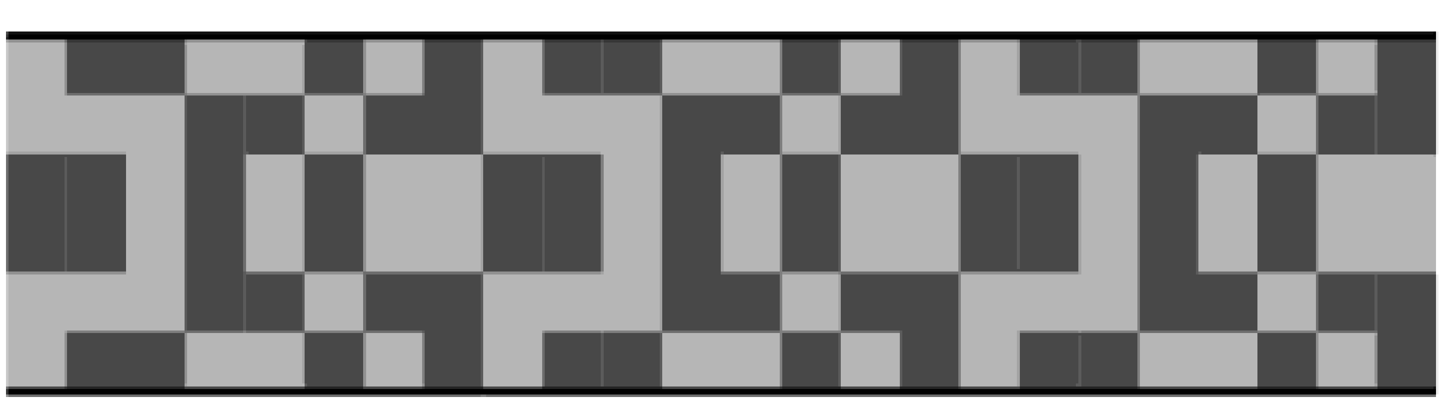
Figure 8.5 (continued)



p: *pm'11*

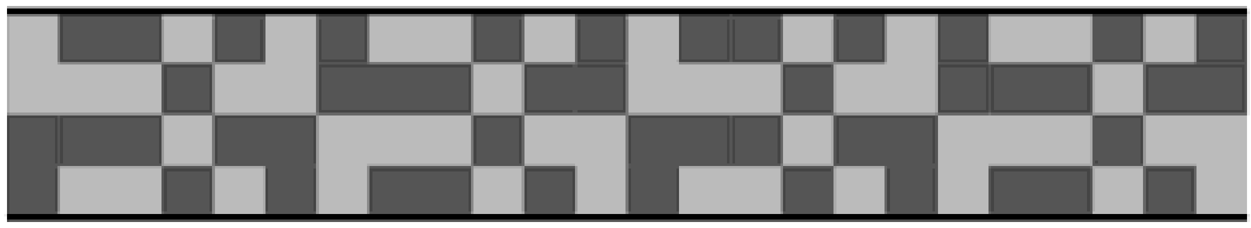


q: *p'1m1*

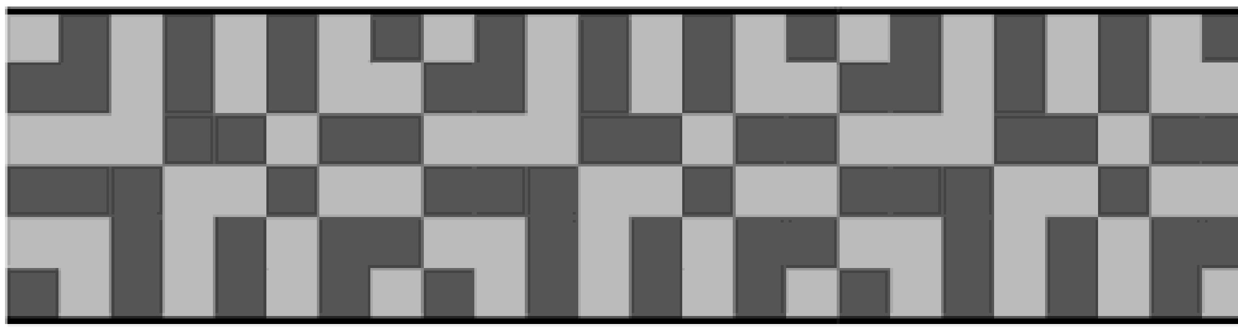


r: *p1m1*

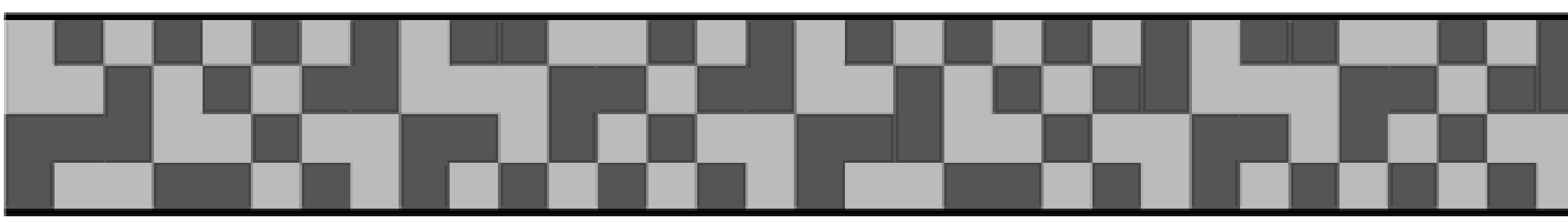
Figure 8.5 (continued)



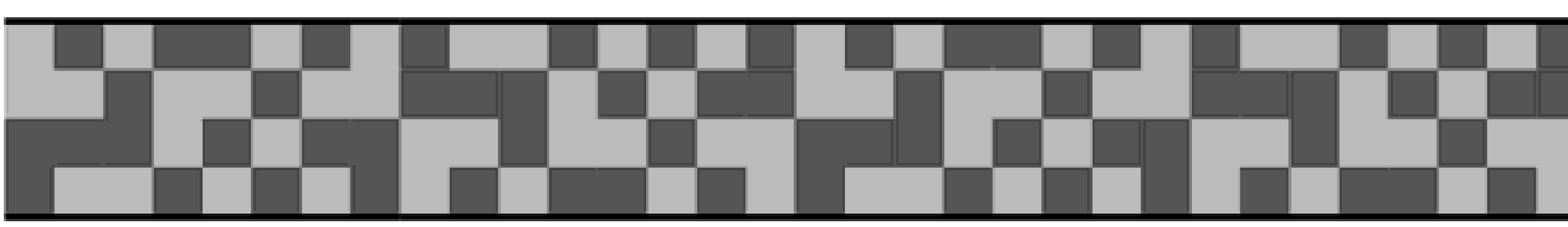
s: *p'1a1*



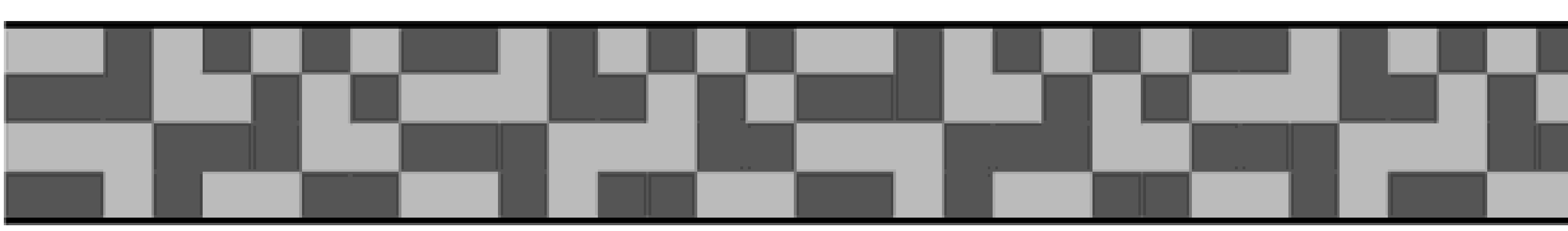
t: *p1m'1*



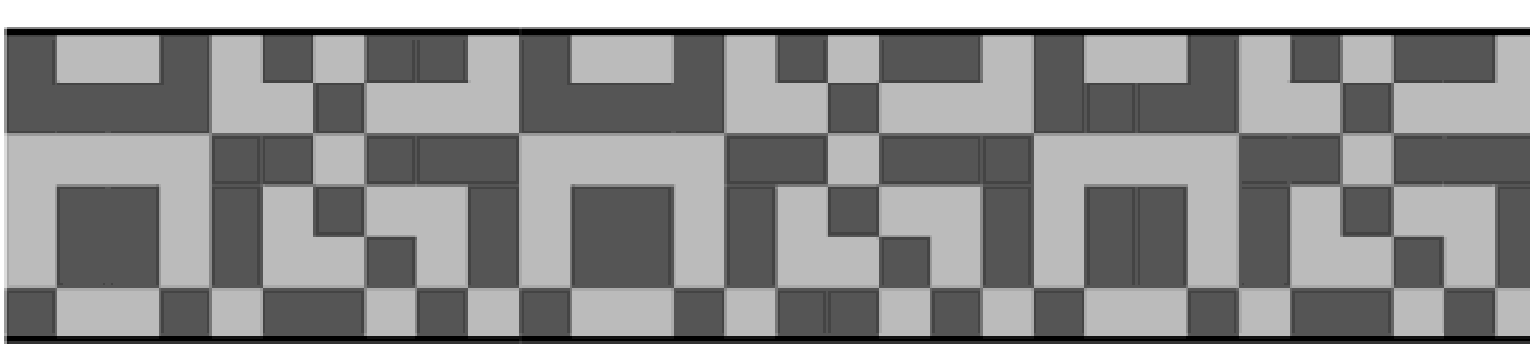
u: *p1a'1*



v: *p1a1*

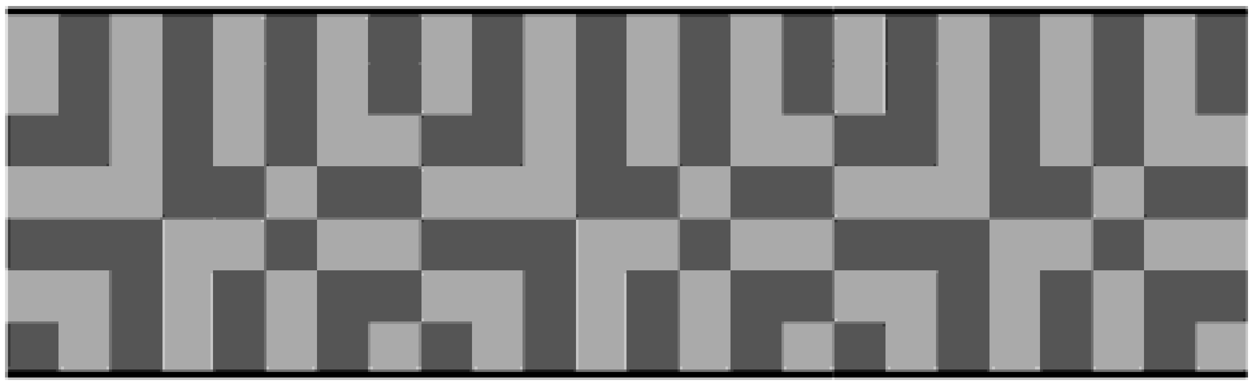


w: *p'111*



x: *p111*

Figure 8.5 (continued)



x: *p111*

Figure 8.5 (conclusion)

8.2 Two-dimensional Lunda-patterns

Consider an infinite grid, IG, having as points (2p, 2q), where *p* and *q* denote arbitrary whole numbers.

A **two-dimensional Lunda-pattern** may be defined as a two-dimensional black-and-white plane pattern with the following characteristic:

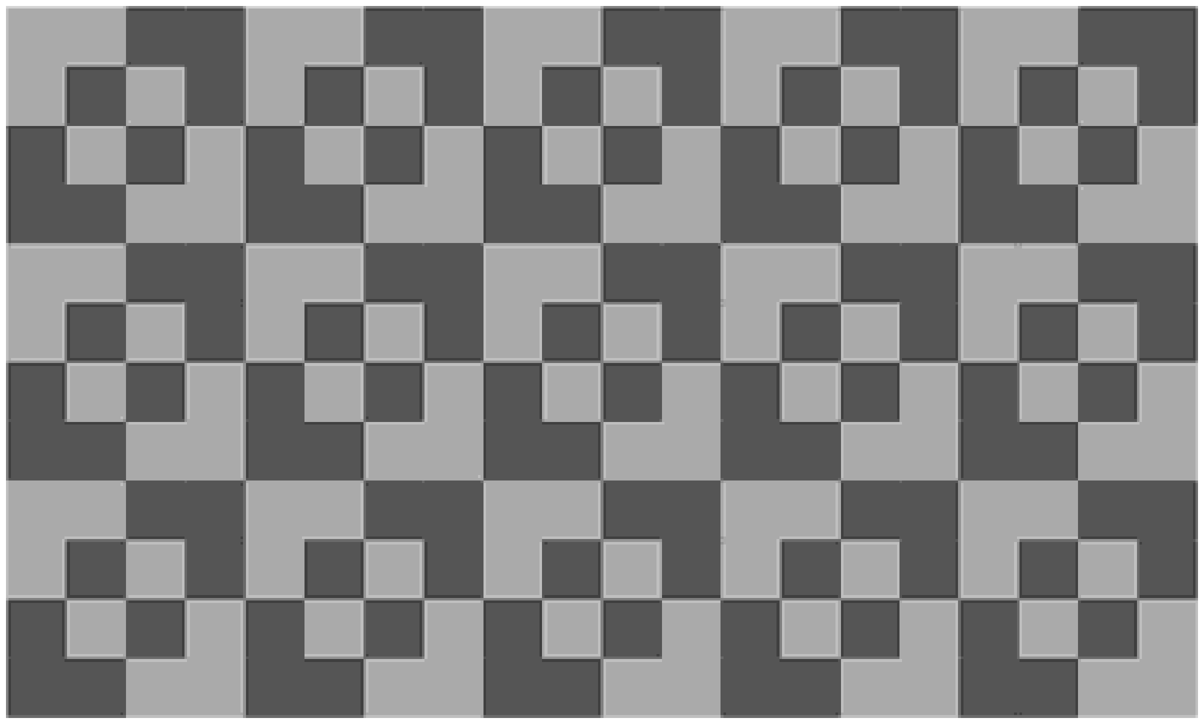
(i) Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are always black and two are white.

Two-dimensional or periodic pattern means that the design admits translations in two or more directions. A two-dimensional pattern is called a two-color pattern if there is some rigid motion of the plane, which interchanges colors everywhere. Two-color, two-dimensional patterns are also called mosaic or tiling.

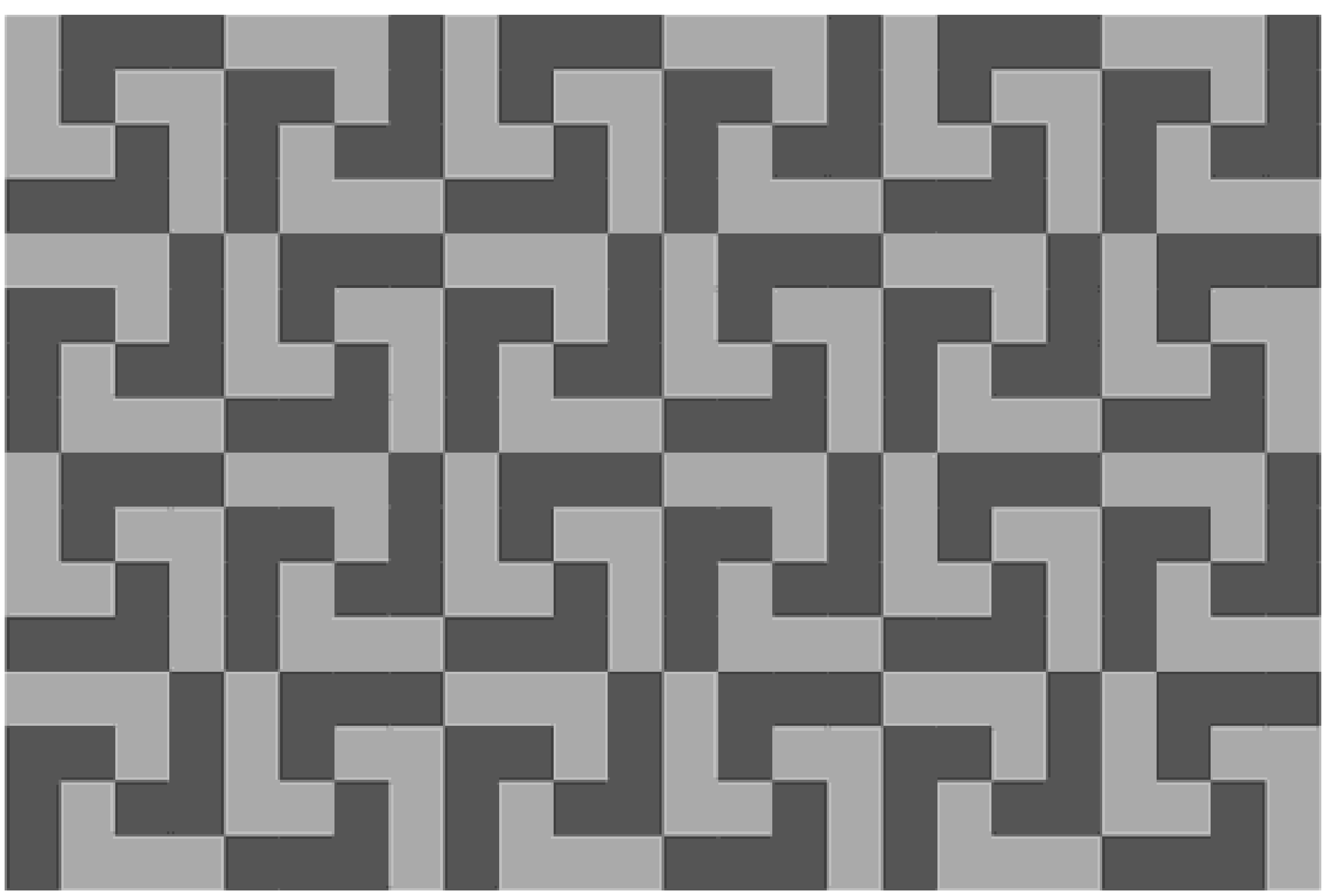
Classification by symmetry results in seventeen one-color and forty-six two-color, two-dimensional pattern classes. Woods (1936) was the first to illustrate all the forty-six classes of mosaics. His mosaics are reproduced in Washburn & Crowe (1988, 74-75). It is interesting to note that two of his patterns (33 and 38) are also Lunda-patterns. They are shown in Figure 8.6.

Washburn & Crowe produced flow charts, which facilitate the classification of one-color and two-color, two-dimensional patterns (1988, pp. 128-131, 140-141, 154-155, 160, 162).

It is impossible to construct Lunda-patterns for each of the classes of two-dimensional patterns, as by definition, two-dimensional Lunda-patterns cannot admit 60^{o} and 120^{o} rotations.



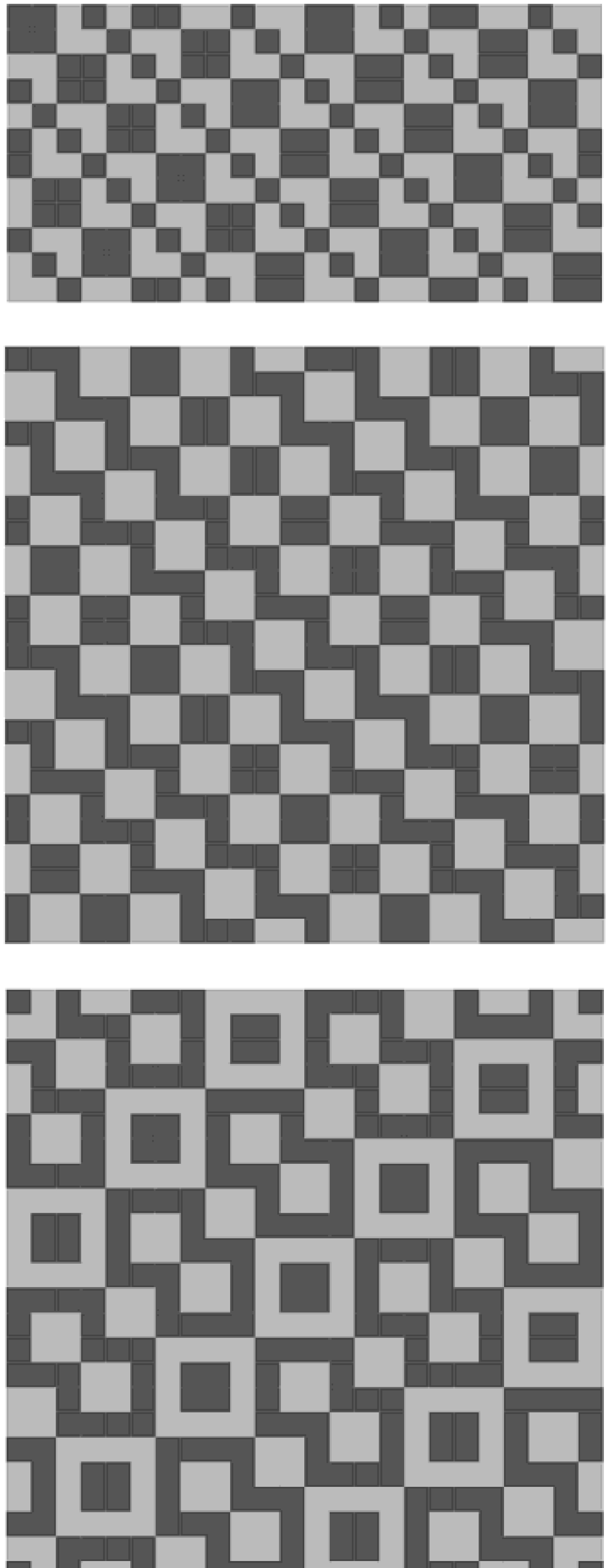
a: *p4'm'm*



b: *p4'gm'*

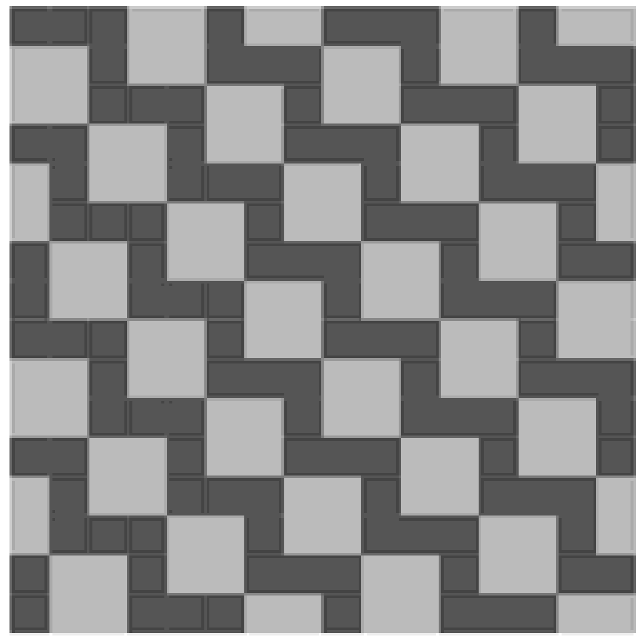
Figure 8.6

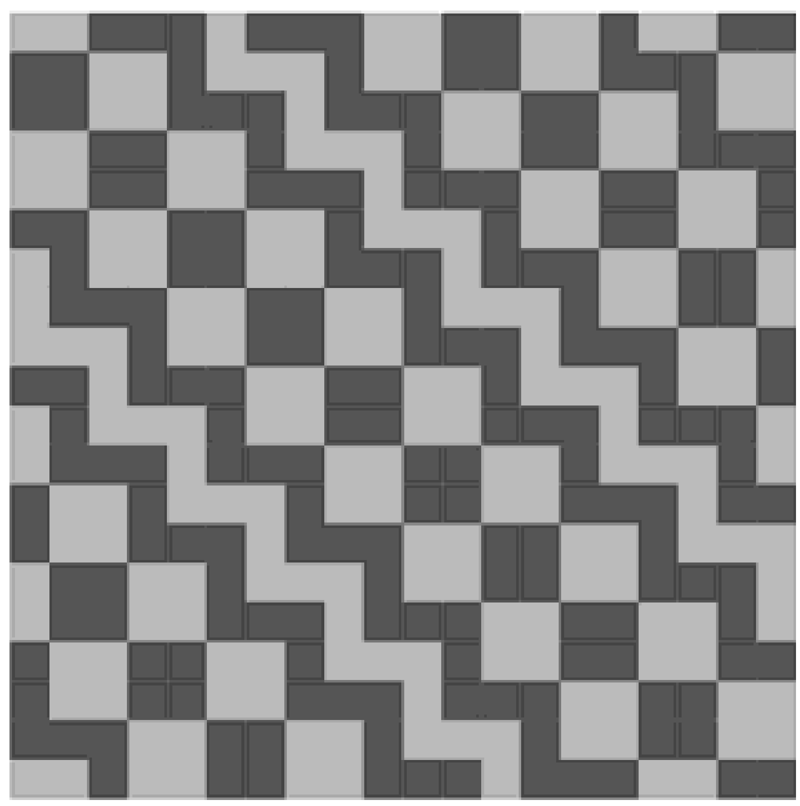
In the following pages we will present examples of two-dimensional Lunda-patterns. Figures 8.7 displays one-color, two-dimensional Lunda-patterns, which admit 180^{o} and 90^{o} rotations respectively. Figures 8.8 and 8.9 show two-color, two-dimensional Lunda-patterns, which admit 180^{o} and 90^{o} rotations consistent with color respectively.



a: *pmm*

Figure 8.7 (1st part)





b: *cmm*

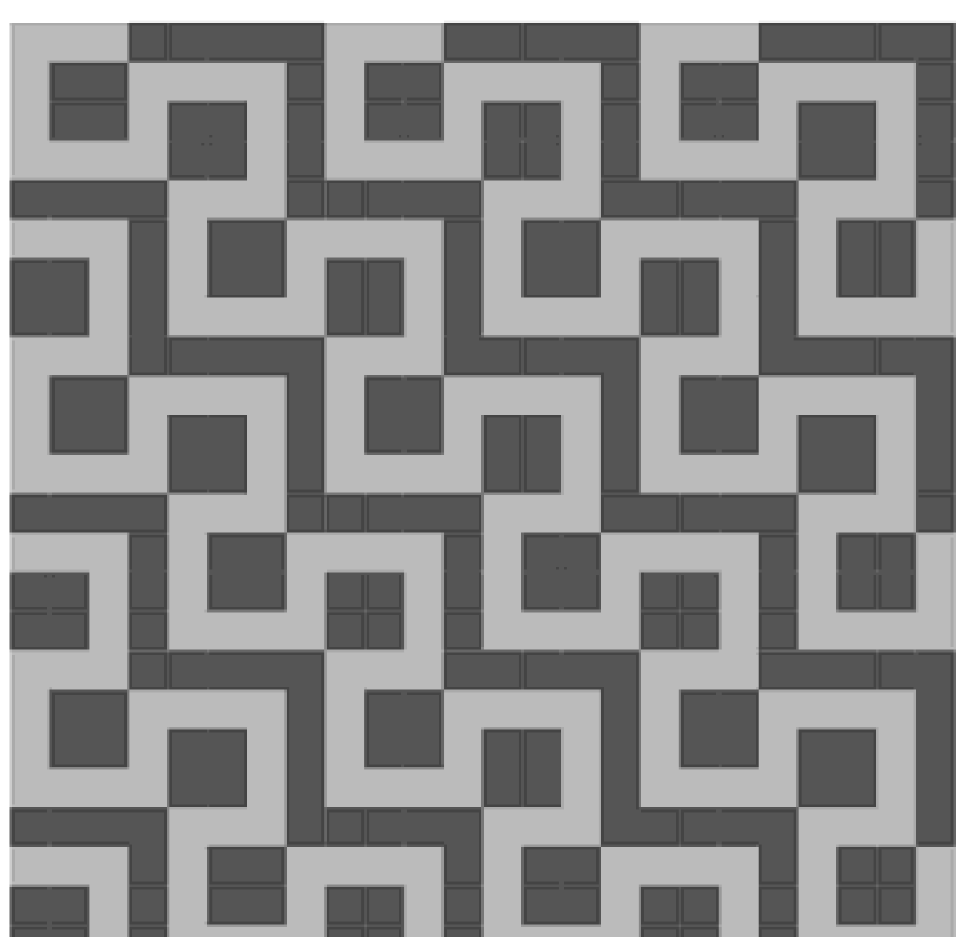
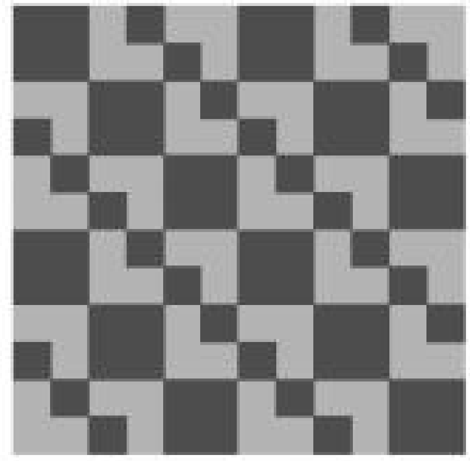
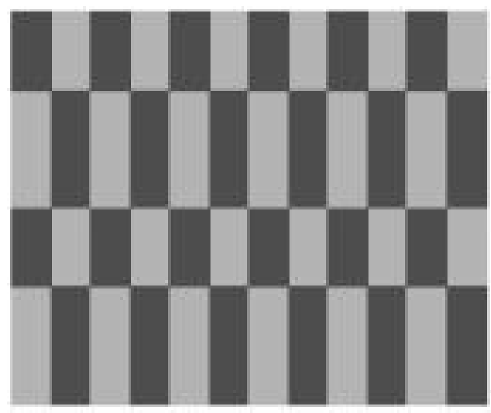


Figure 8.7 (continued)

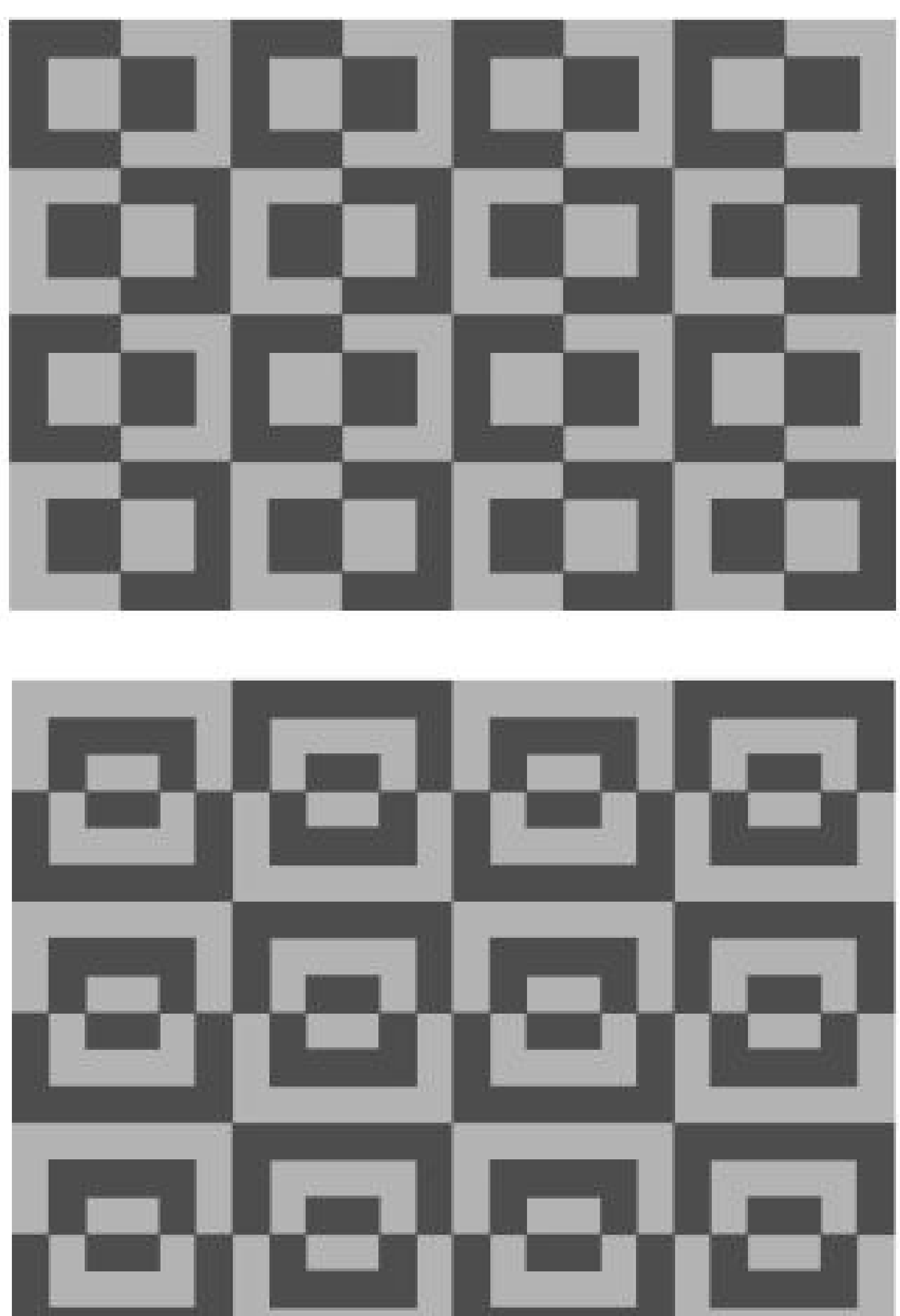


c: *pmg*

Figure 8.7 (conclusion)

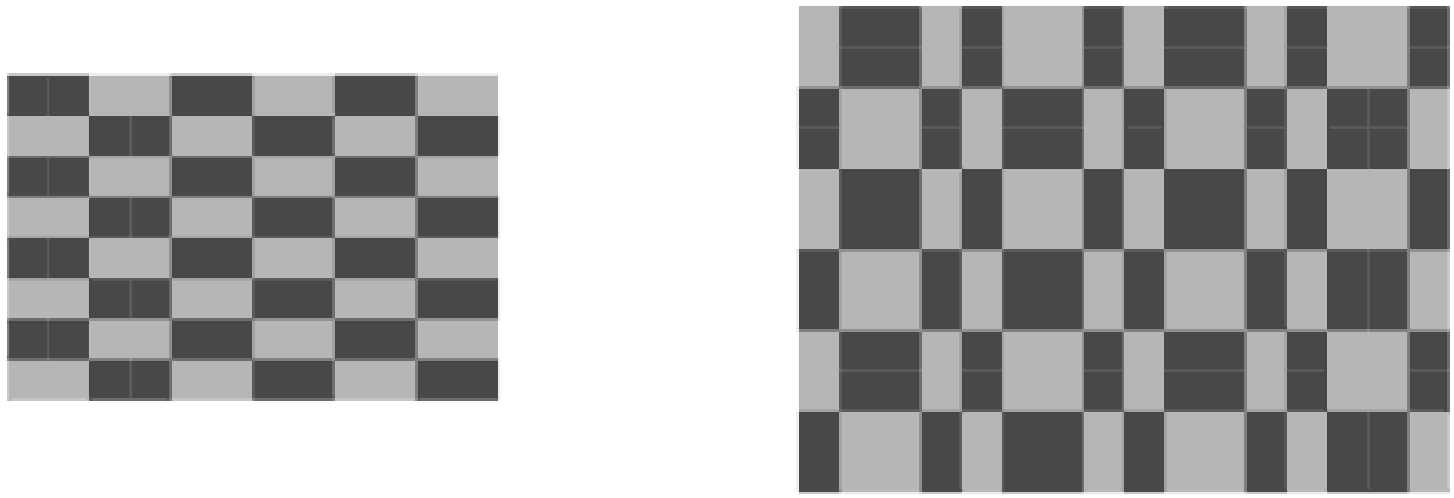


a: p_b*'mm*

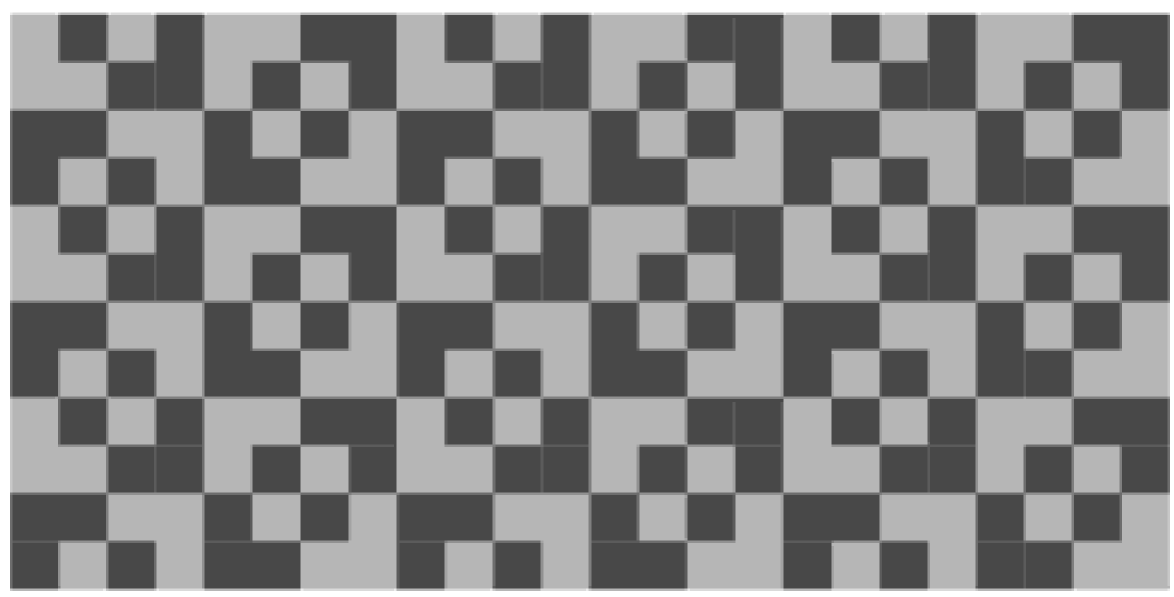


b: p_b*'gm*

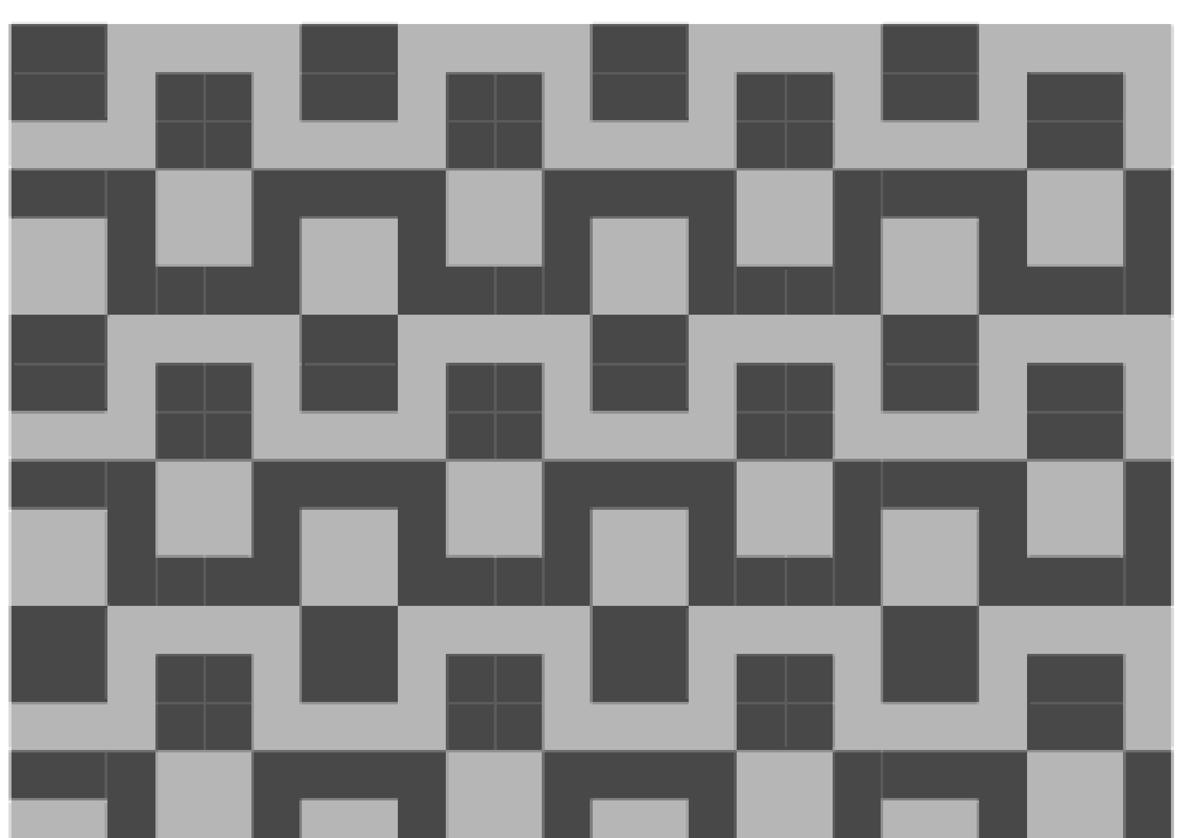
Figure 8.8 (1st part)



c: *c'mm*

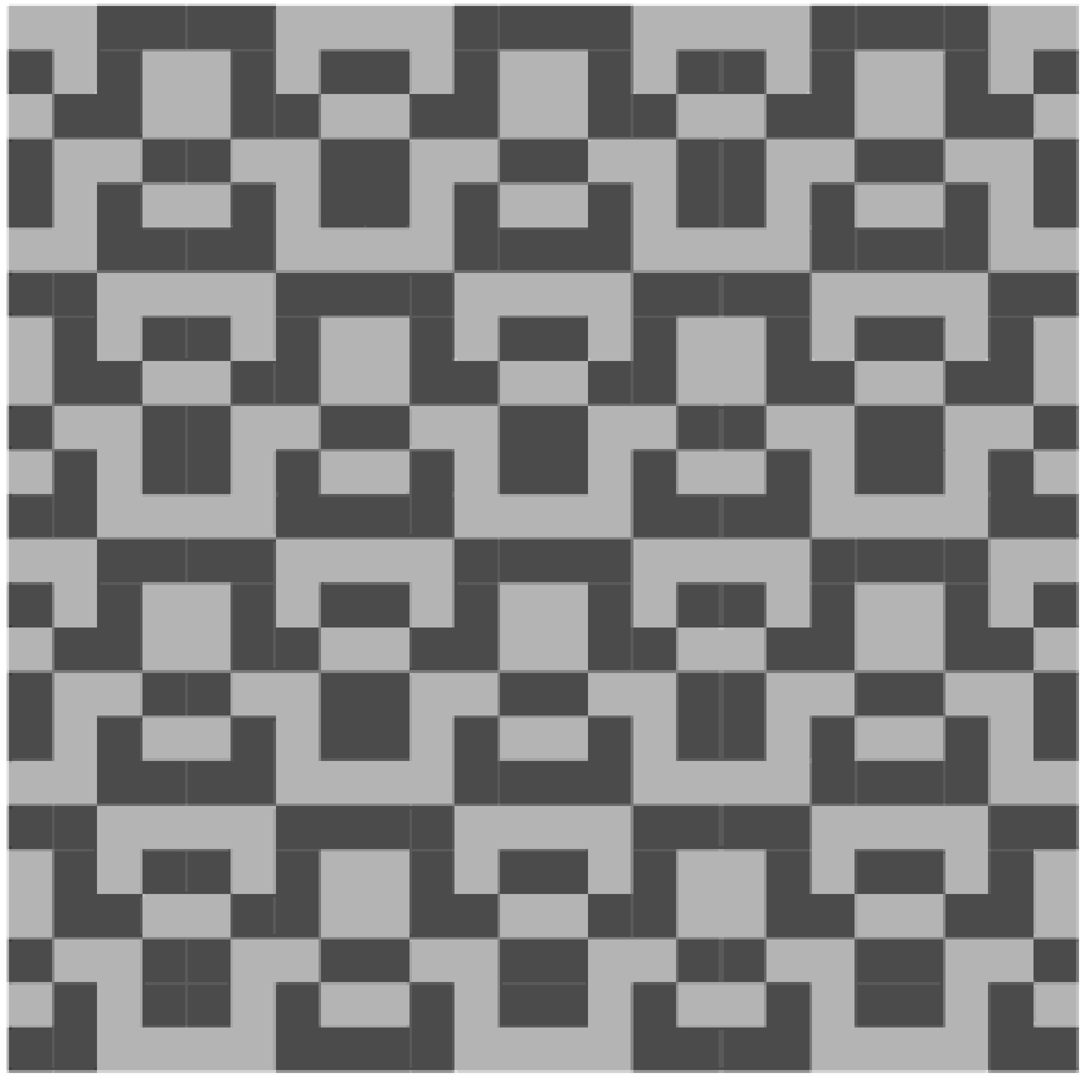


d: p_c*'gg*

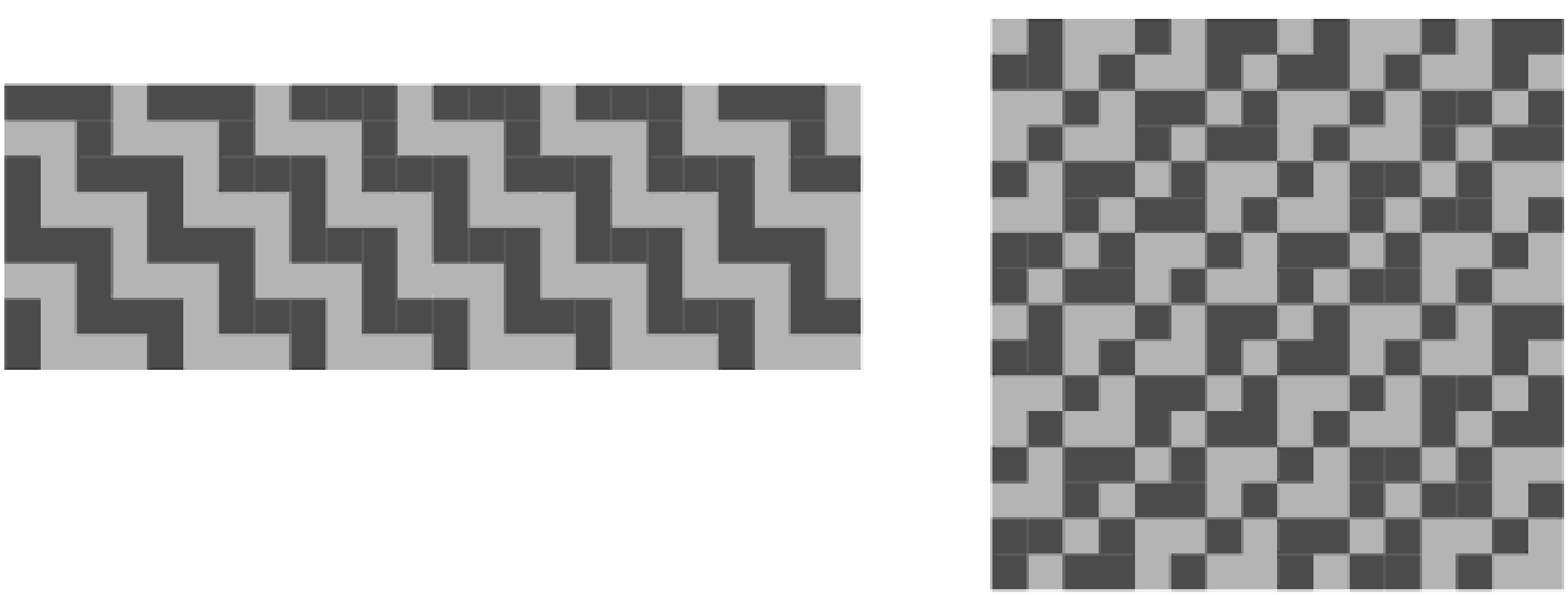


e: p_c*'mg*

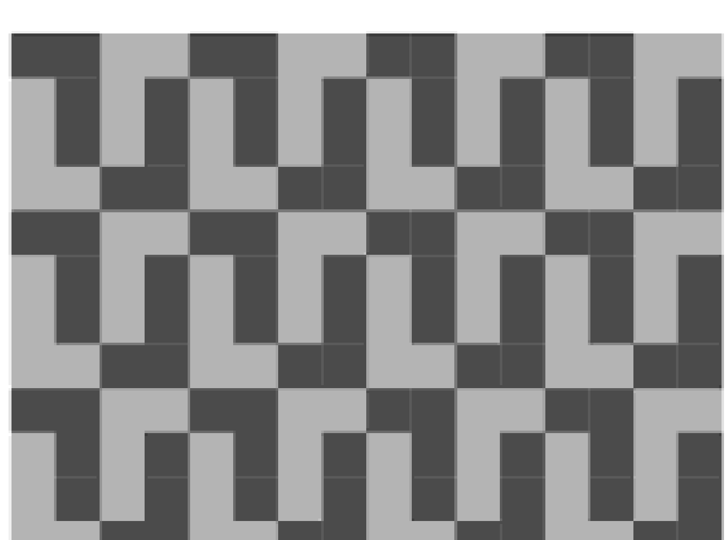
Figure 8.8 (continued)



f: *cmm'*

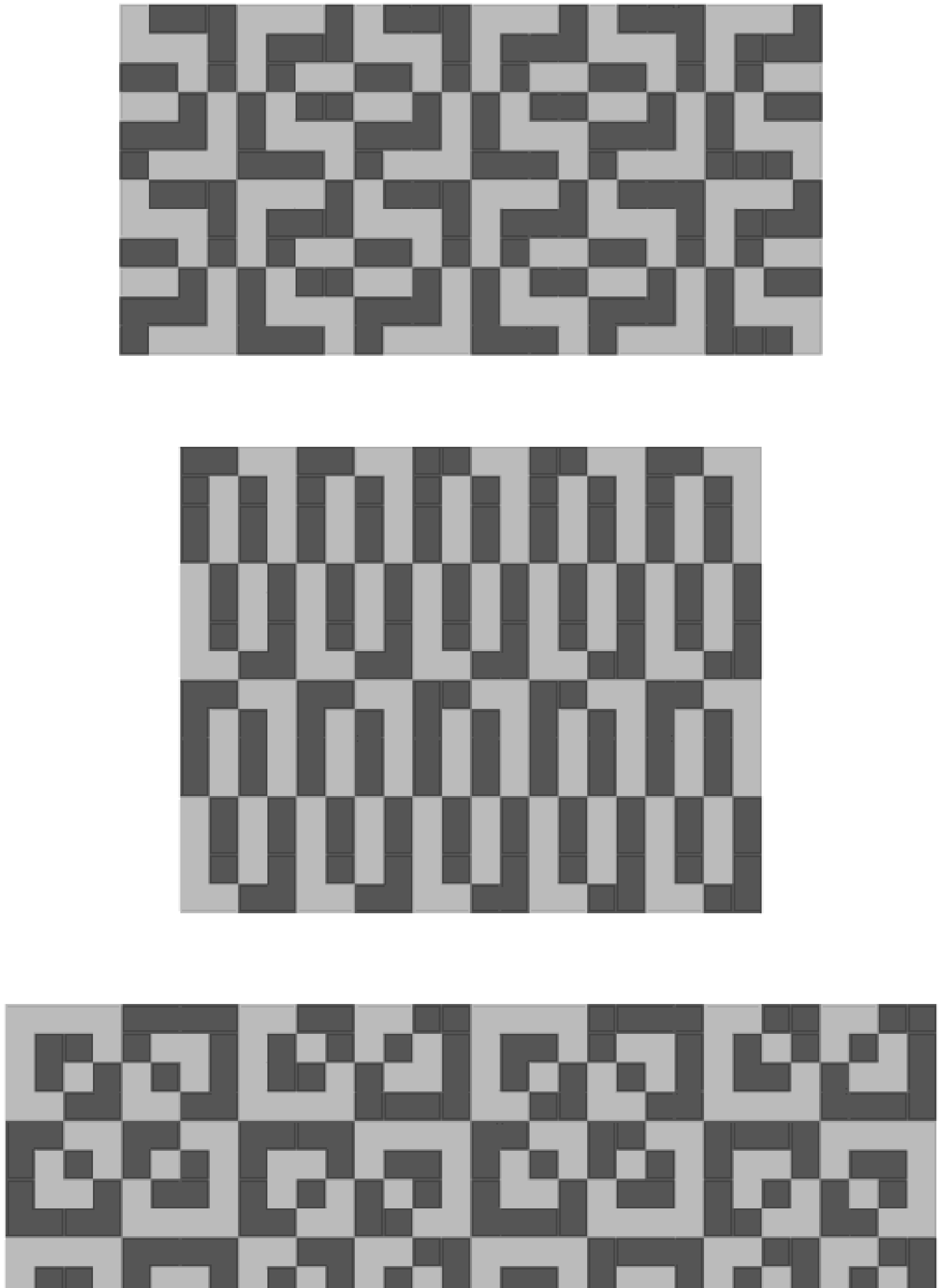


g: p_b*'mg*



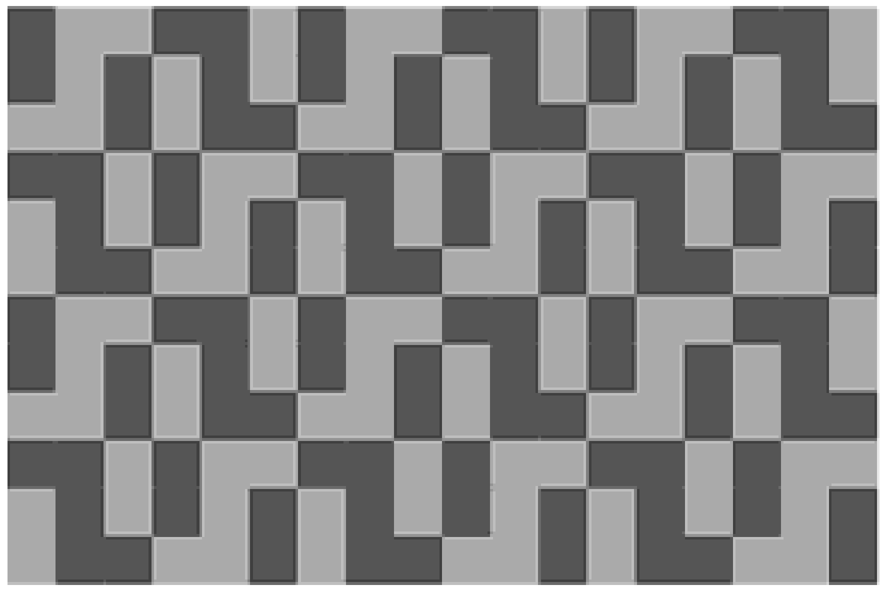
h: *pm'g*

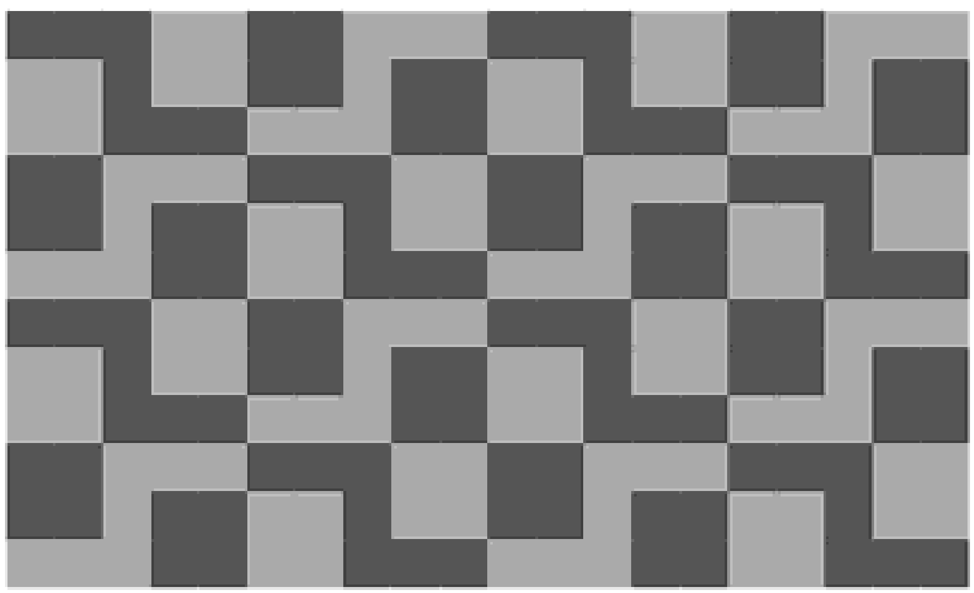
Figure 8.8 (continued)



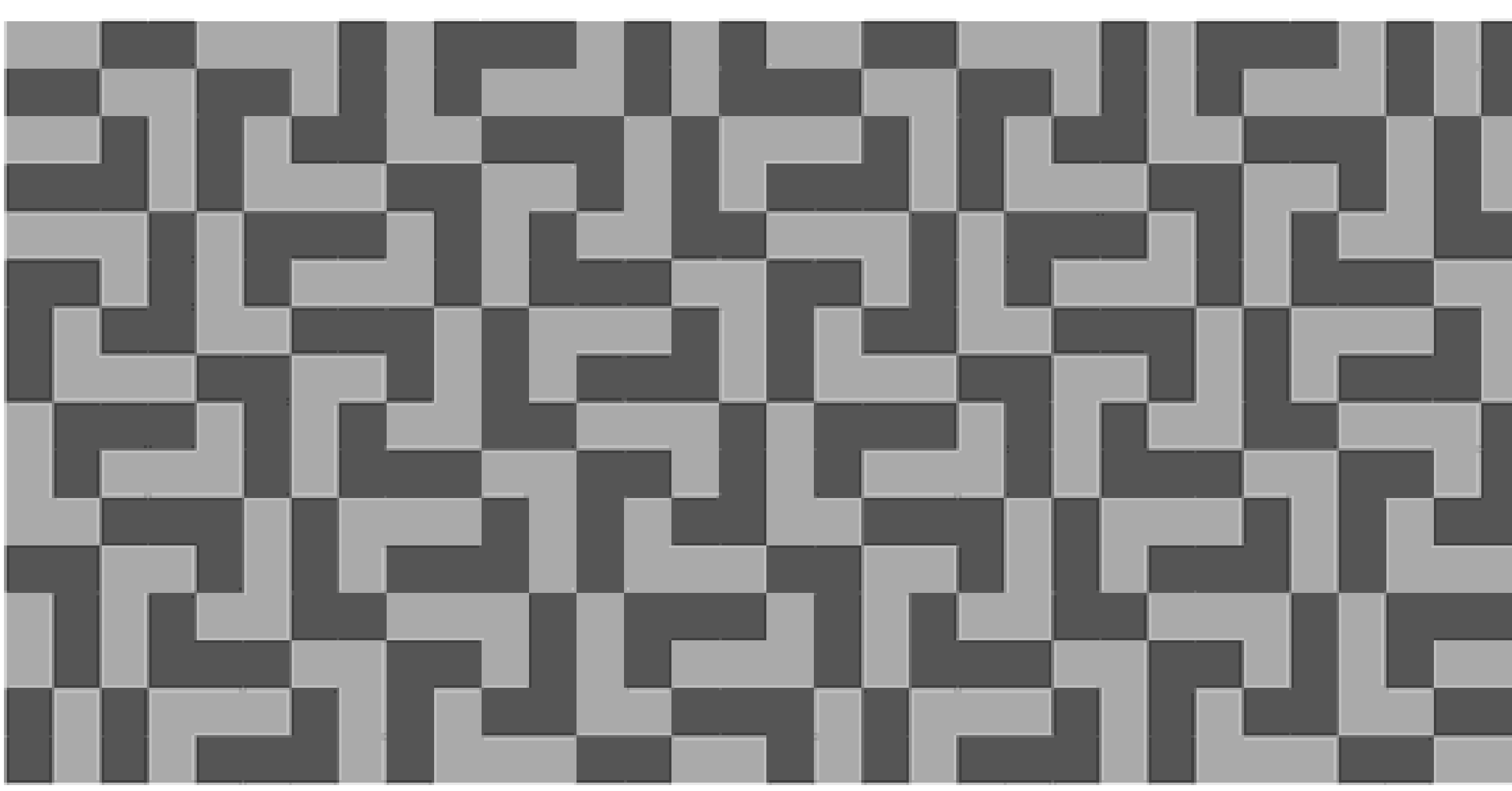
i: *pm'm'*

Figure 8.8 (continued)



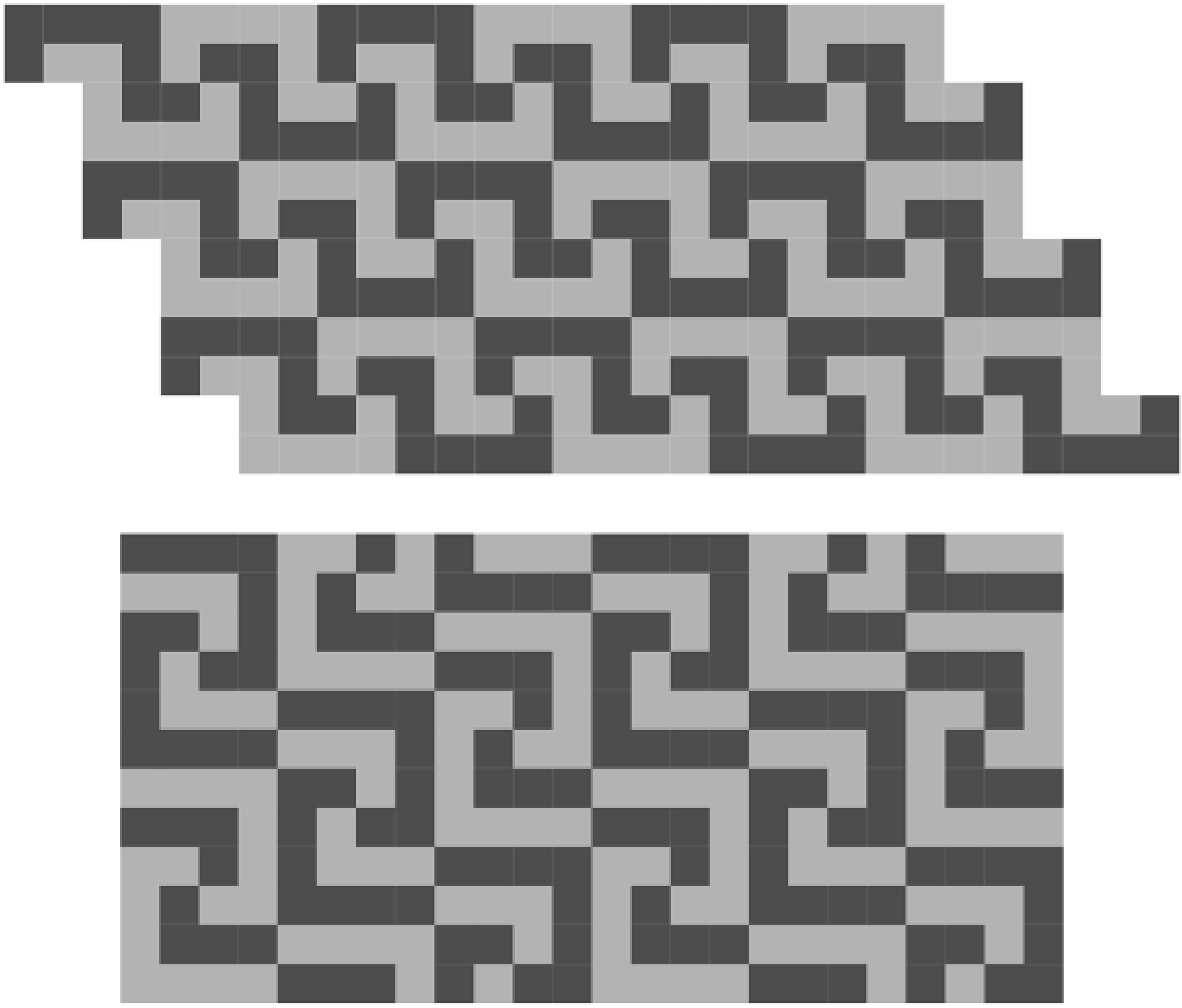


j: *cm'm'*



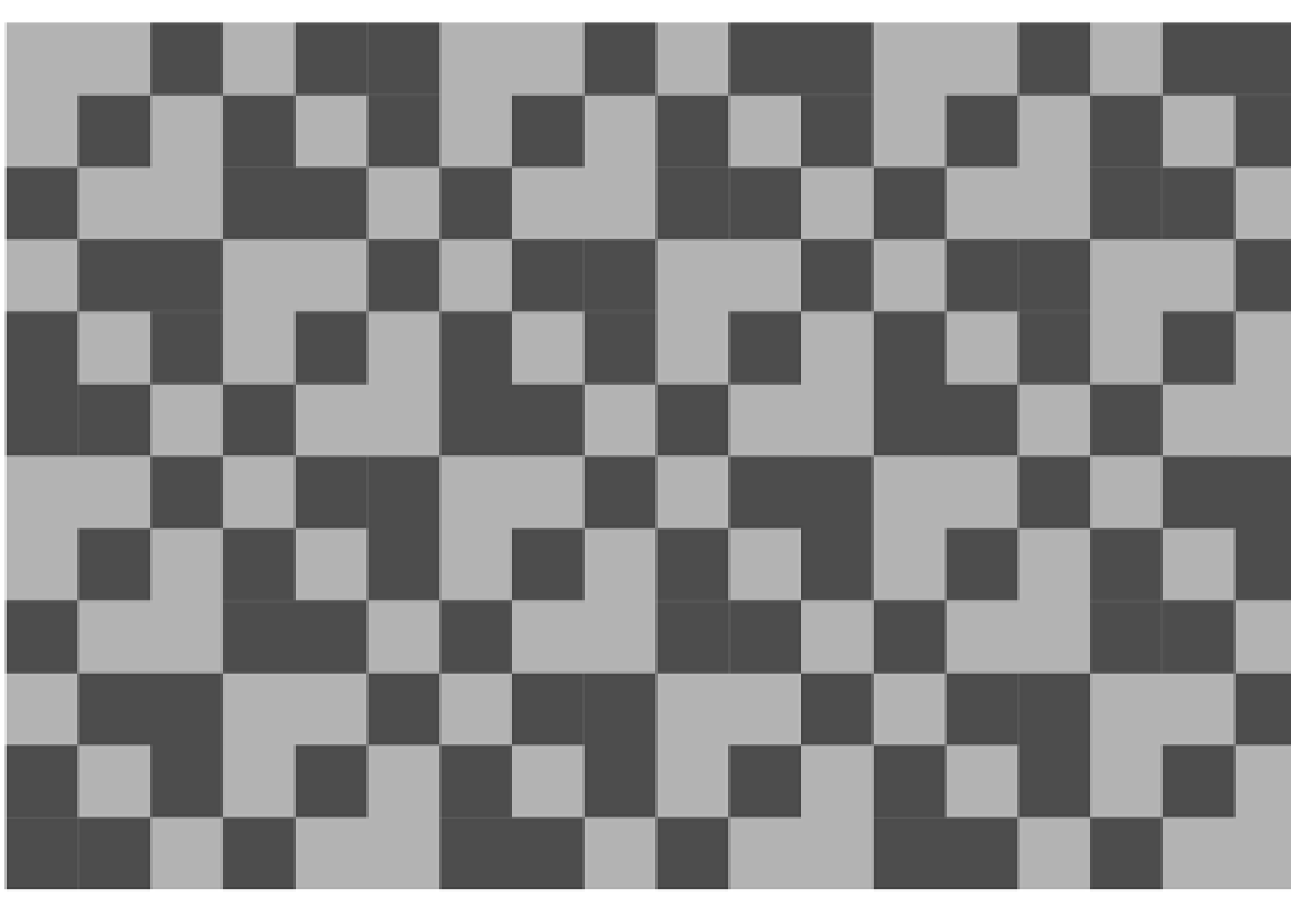
k: *pg'g'*

Figure 8.8 (continued)



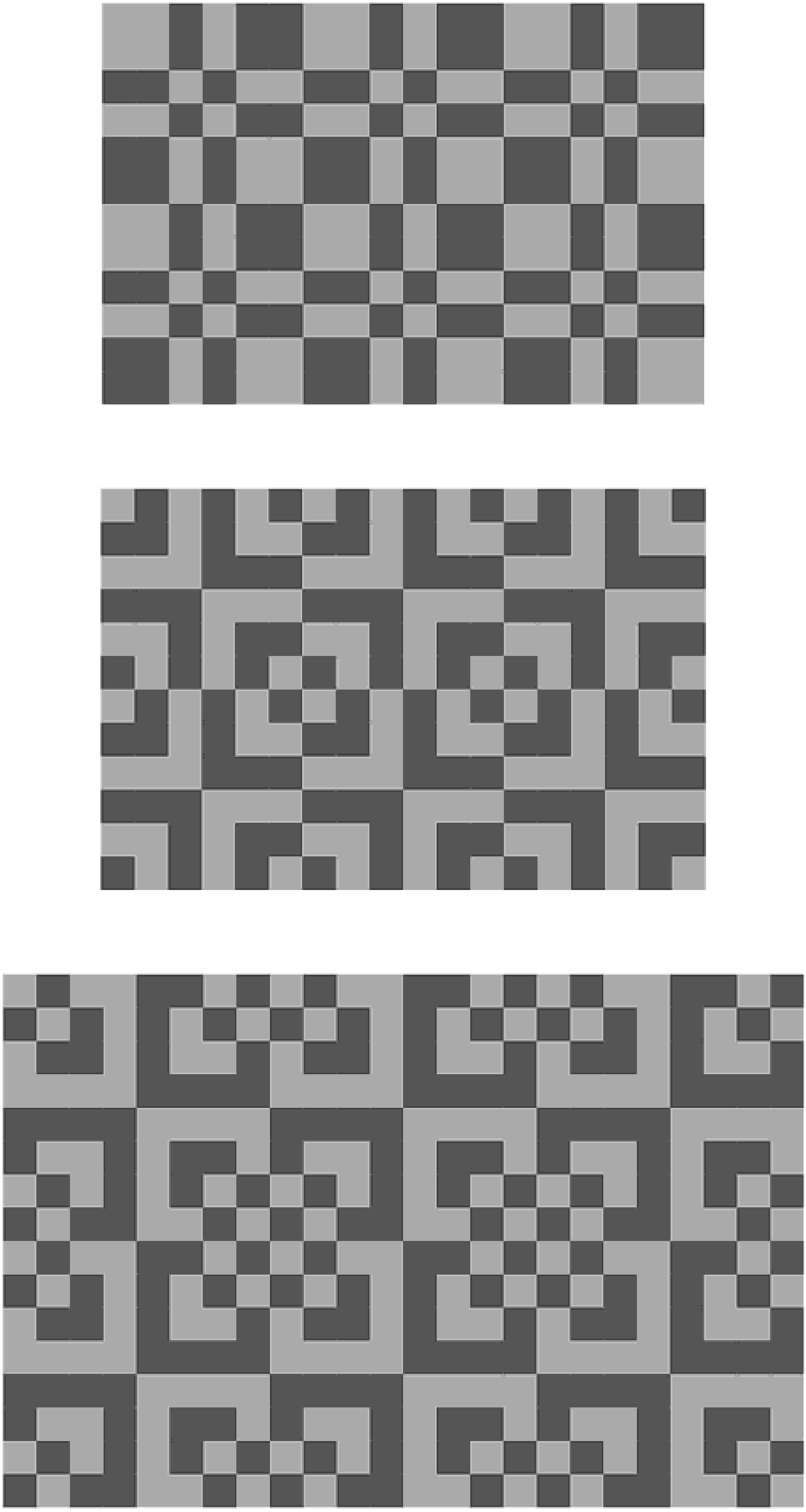
l: p_b'2

Figure 8.8 (conclusion)



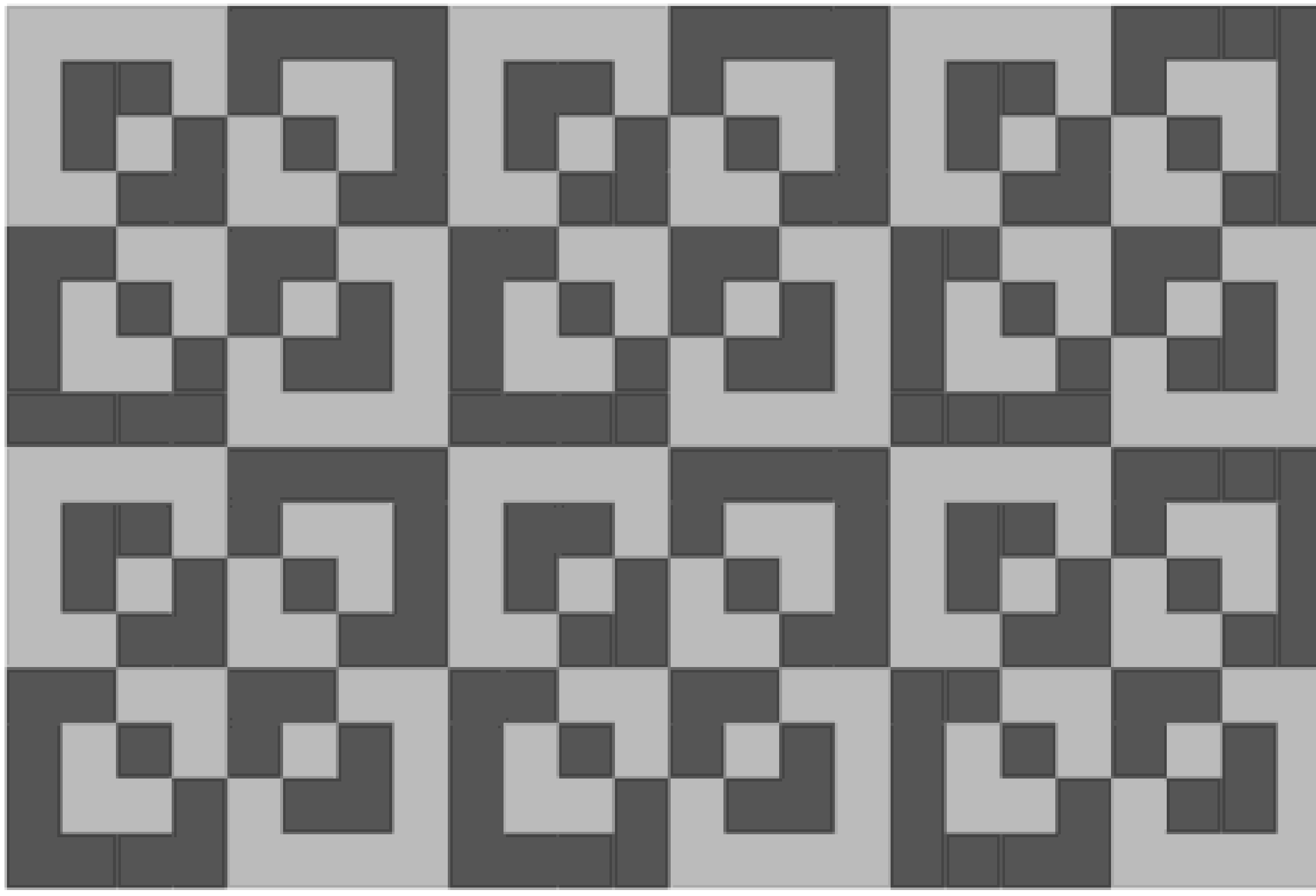
a: *p4'mm'*

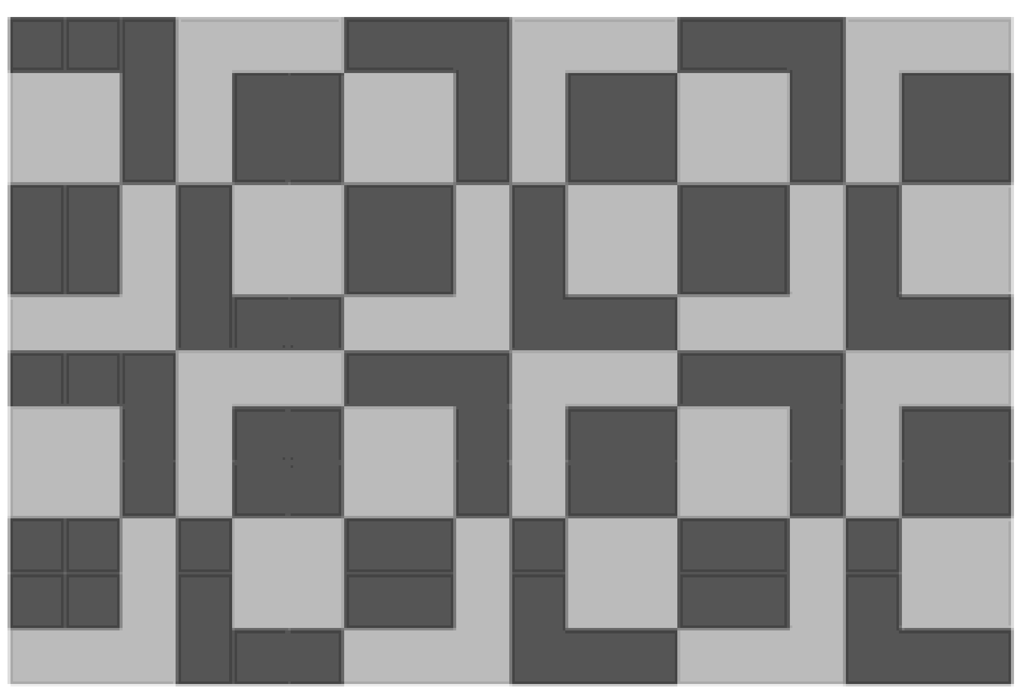
Figure 8.9 (1st part)

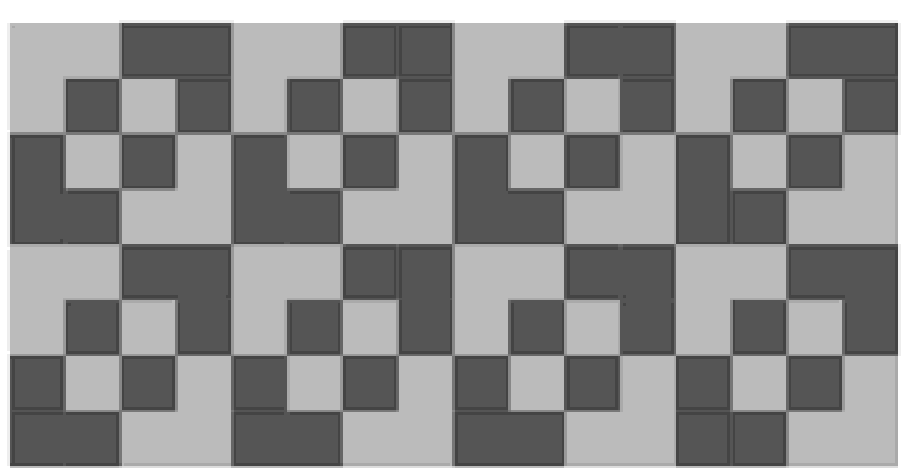


b: *p4'm'm*

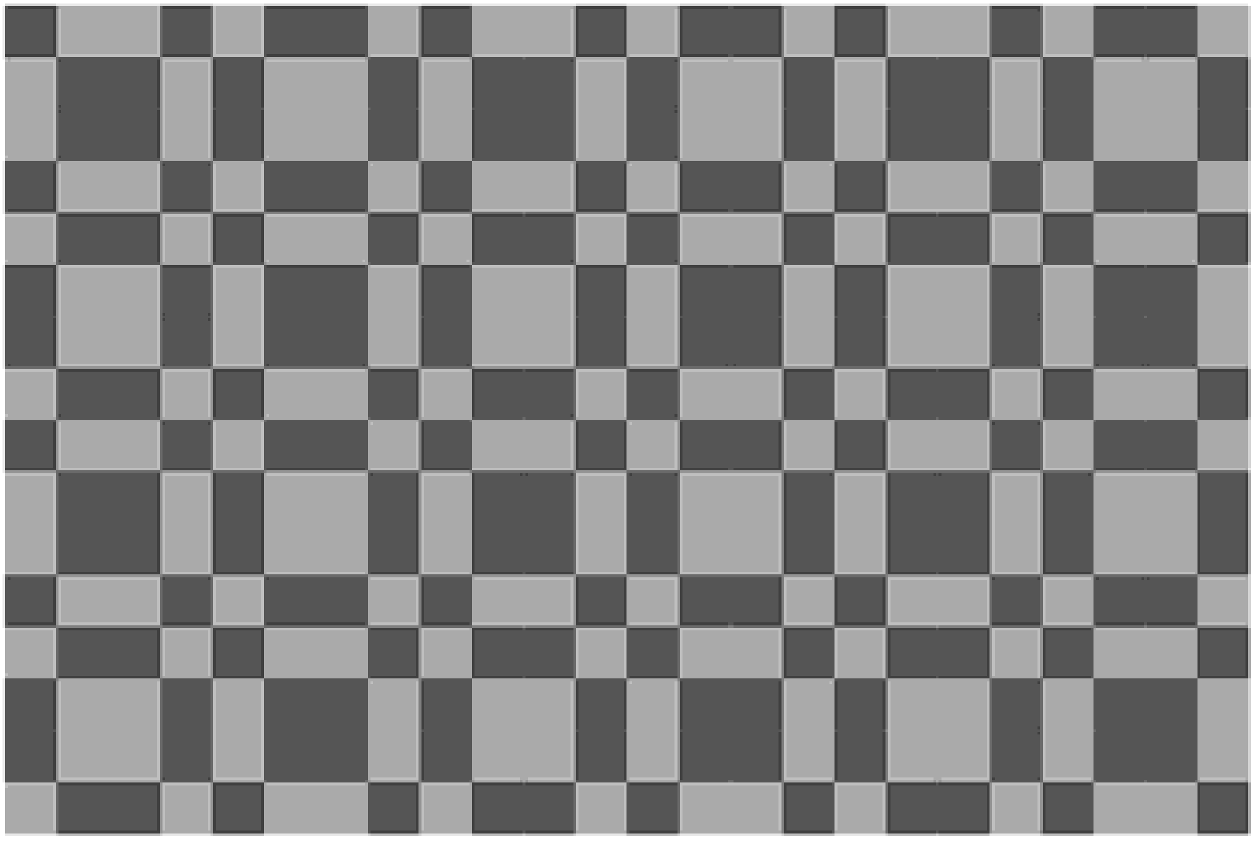
Figure 8.9 (continued)

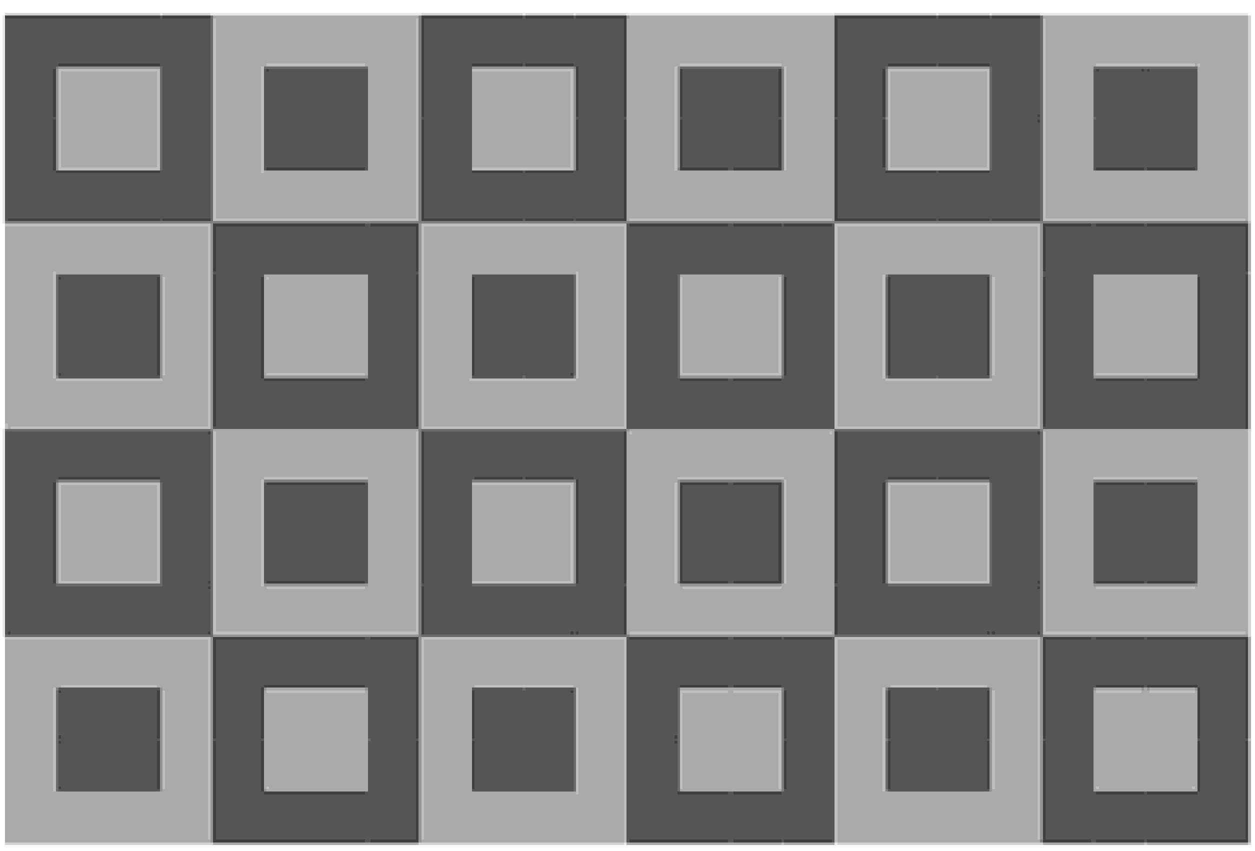






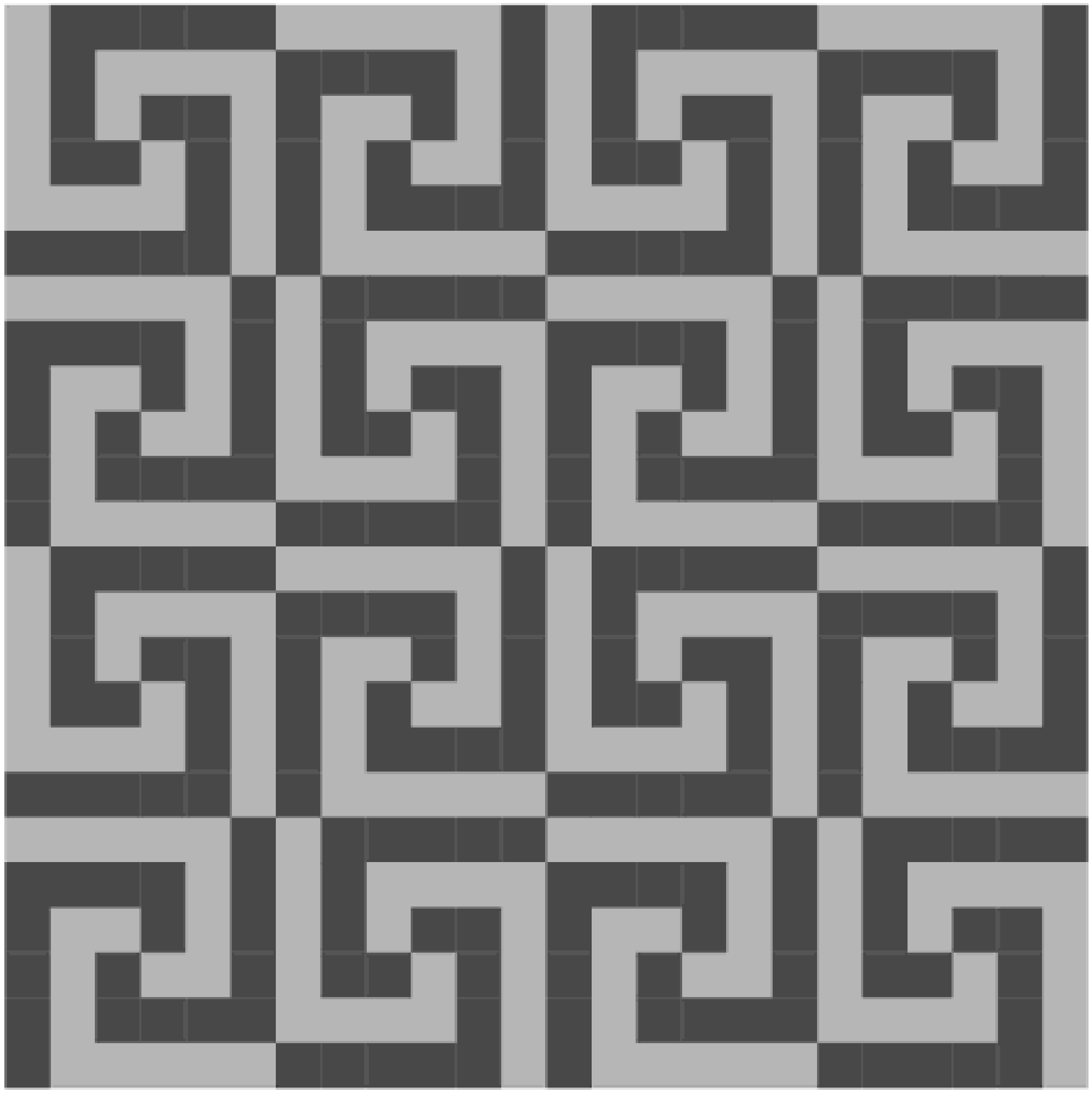
b: *p4'm'm*
Figure 8.9 (continued)



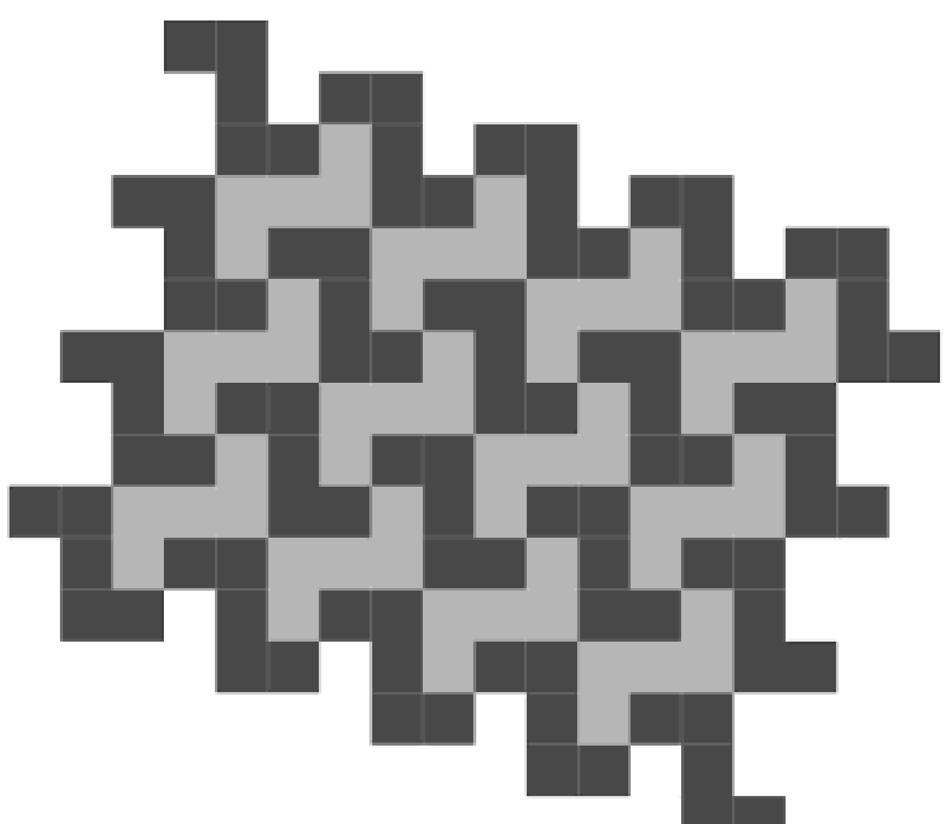


c: p_c'*4mm*

Figure 8.9 (continued)

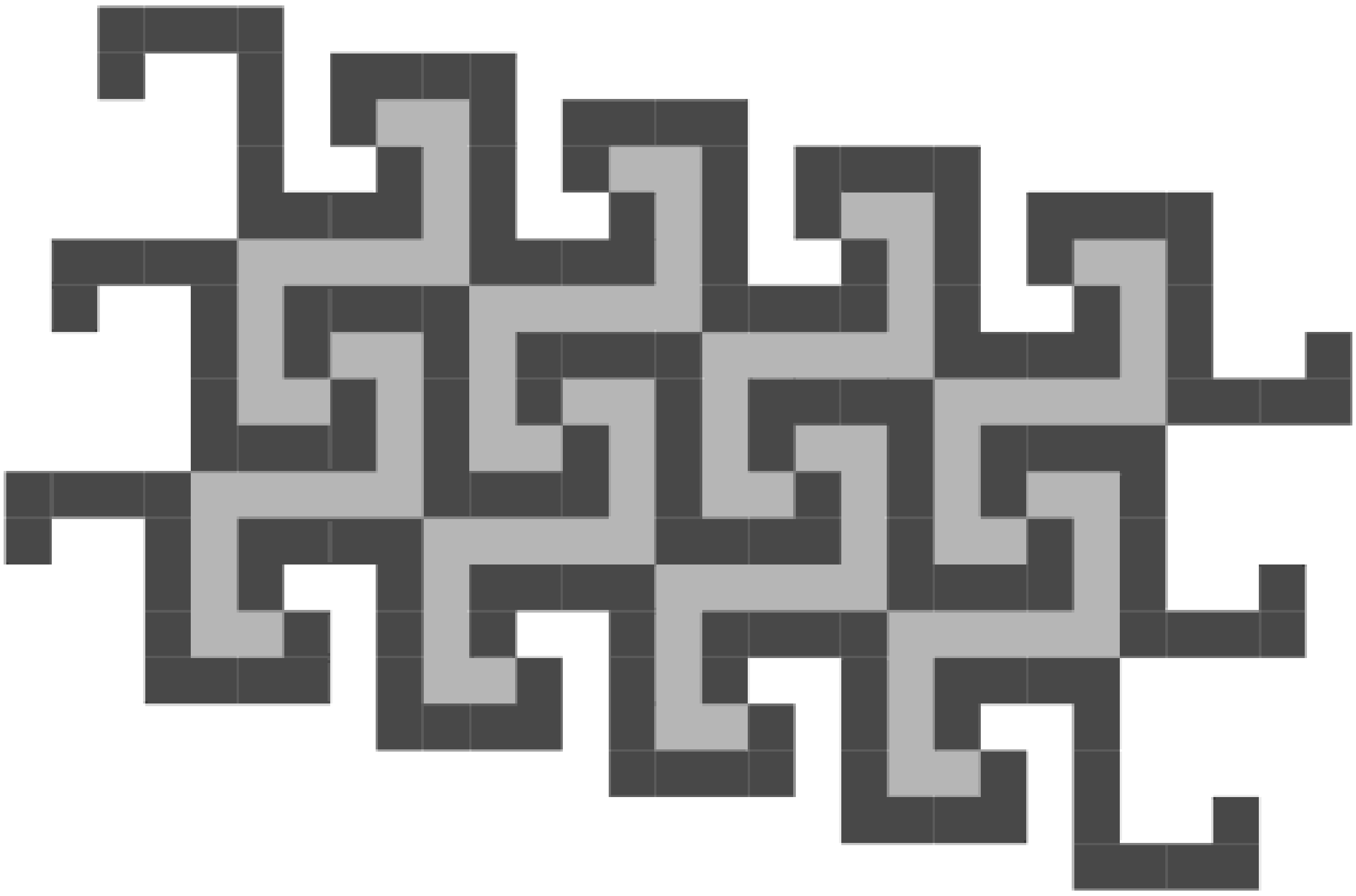


d: *p4'gm'*



e: *p4'*

Figure 8.9 (continued)



e: *p4’*
Figure 8.9 (conclusion)

Reference

Washburn, Dorothy & Crowe, Donald (1988), *Symmetries of Culture. Theory and Practice of Plane Pattern Analysis*, University of Washington Press, Seattle, 299 pp.

Chapter 9

ON THE GEOMETRY OF CELTIC KNOTS AND THEIR LUNDA-DESIGNS [1]

George Bain (1951), his son Iain Bain (1986), Aidan Meehan (1991) and Peter Cromwell (1993) analyzed the construction methods of the beautiful Celtic knot works that illuminate the pages of the Book of Kells and the Lindisfarne gospels and decorate Pictish metalwork and stone crosses in the British Isles (8th and 9th centuries AD). Together with the Celtic spirals and key patterns these knot works may be interpreted, in the words of the late John Fauvel (1990, p.6), as *Celtic ethnomathematics*. Harald Gropp (1996) draws attention to calendar reckoning as part of Celtic mathematics. In this chapter, I will present examples of Celtic knots and show how they generate attractive black-and-white designs that I call Lunda-designs. Global and local symmetry properties of Lunda-designs will be analyzed, as well as suggestions for the educational use of these designs will be given.

Figure 9.1a presents the Celtic foundation knot (Meehan, 1991, p.8). It may be generated in the following way. Consider the 2x2 point grid in Figure 9.1b. Imagine a light ray emitted from A, making an angle of 45° with the sides of the square and being reflected on the sides of the square and on a double-sided mirror between the points B and C (Figure 9.1c). After several reflections the light ray returns to A (Figure 9.1d). Figures 9.1e and 9.1f present the smooth version of the closed polygonal path of the light ray. This version will be called a *mirror curve* (cf. Gerdes, 1990, etc.; Jablan, 1995). The Celtic

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[1] First published in: *Mathematics in School* (UK), Vol. 28, No. 3, May 1999, 29-33.

foundation knot is topologically equivalent to the mirror curve in Figure 9.1f: the lower zigzag loop has been smoothened into an arc. Many Celtic knots can be generated in a similar way.

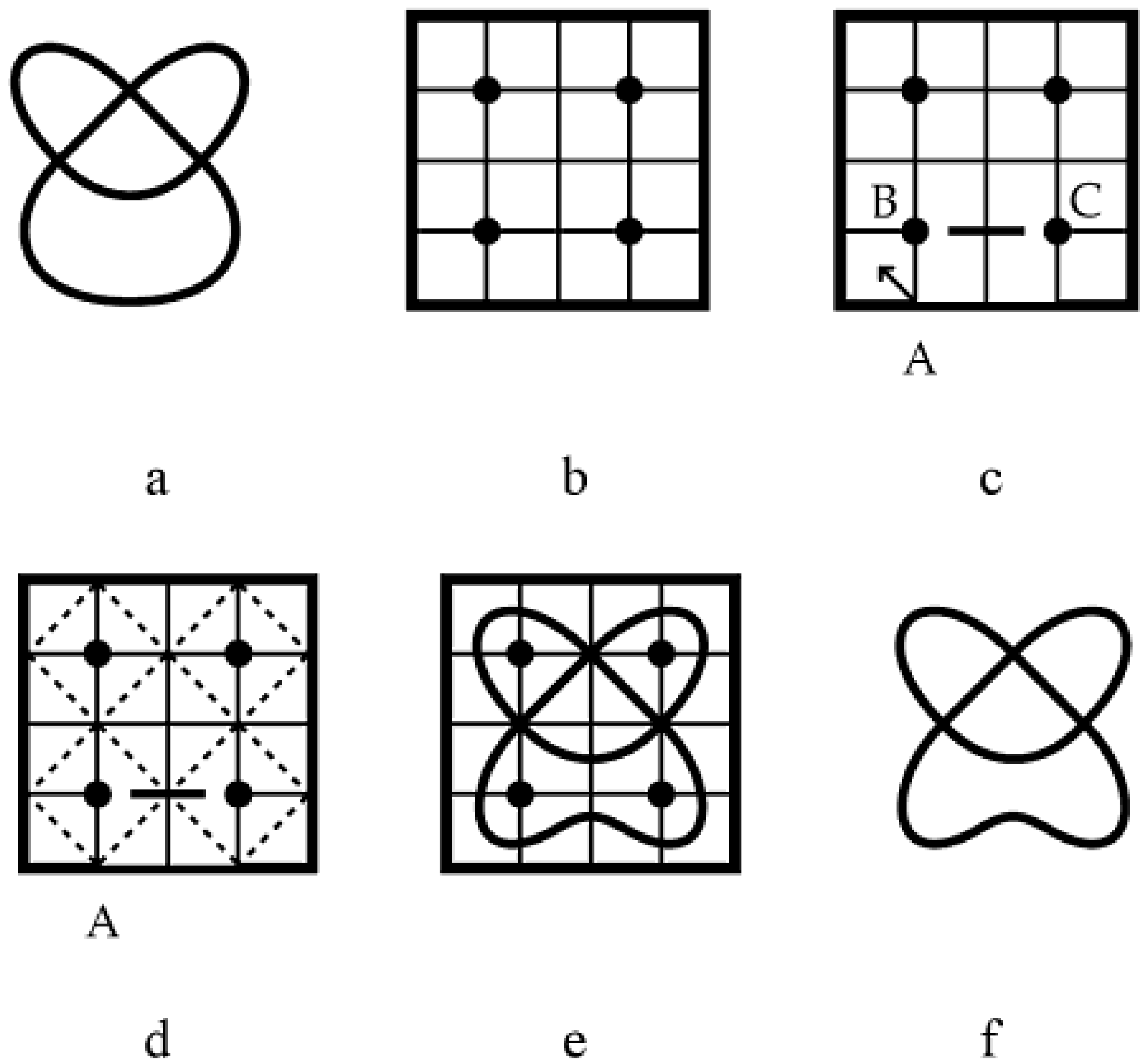
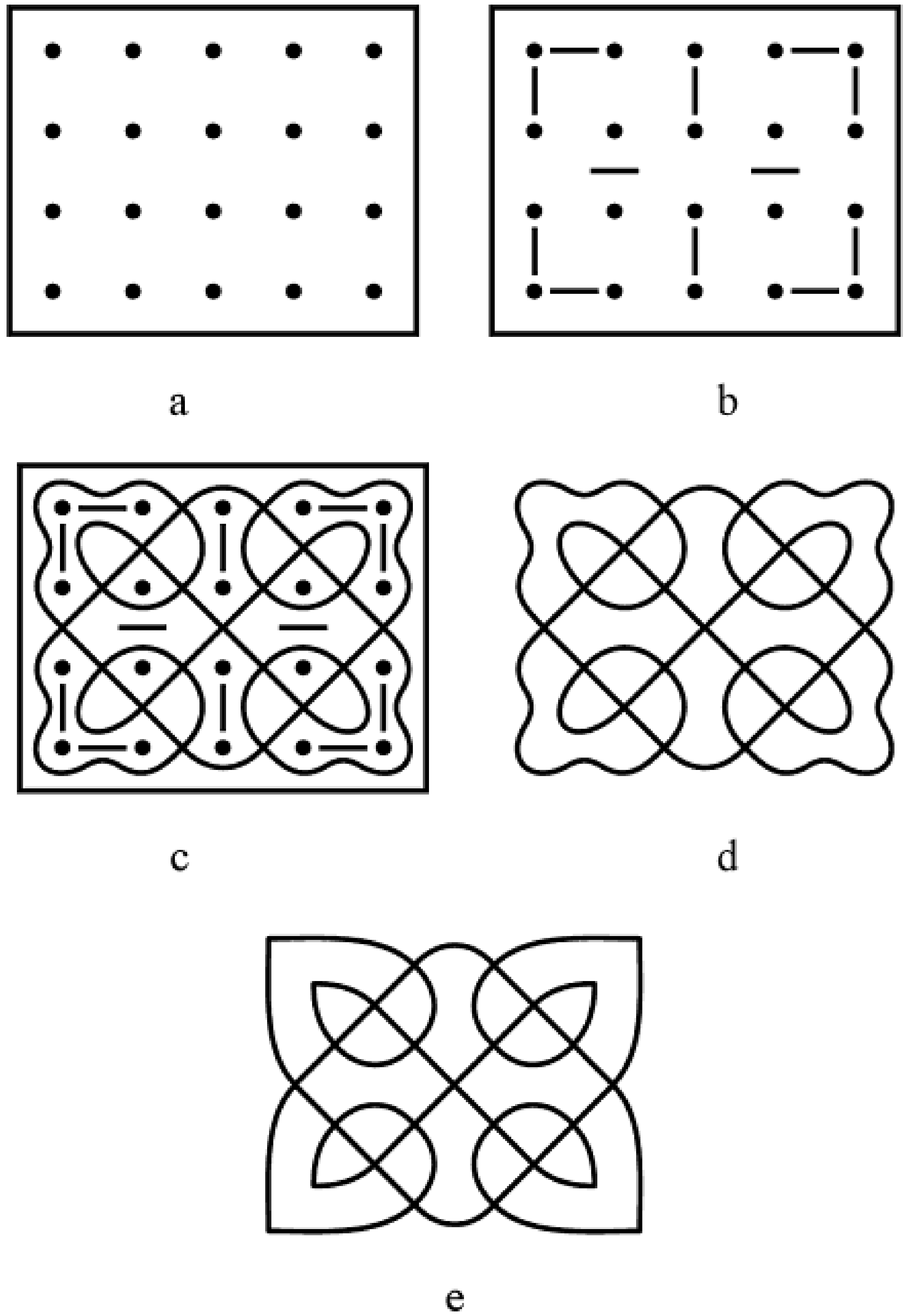


Figure 9.1

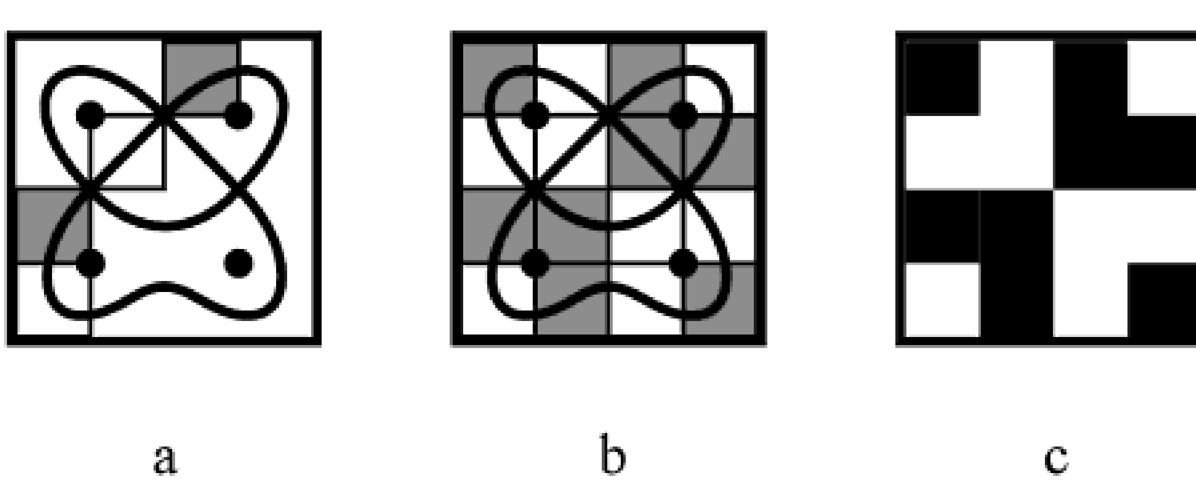
Figure 9.2 presents a second example: Consider a 4x5 point grid (Figure 9.2a). In the centre between some (horizontal or vertical) neighboring grid points double sided mirrors are placed horizontally or vertically (Figure 9.2b). Figure 9.2c shows the subsequent mirror curve, that is equivalent to the Celtic knot in Figure 9.2d, the Lagore Crannog knot (Meehan, 1991, p. 113).

In the two examples the distance between two horizontal or vertical neighboring grid points has been chosen equal to 2 units, and the distance between a border grid point and the rectangular border equal to one unit. By consequence, each of the mirror curves passes exactly once through each of the unit squares in which the respective rectangular grids can be decomposed. This enables us to color the successive unit squares through which the curve passes alternately black and white. Starting with a white unit square, we obtain in the case of the Celtic foundation and Lagore Crannog knots (see Figures 9.3 and 9.4) the black-and-white designs presented in Figures 9.3c and 9.4b.



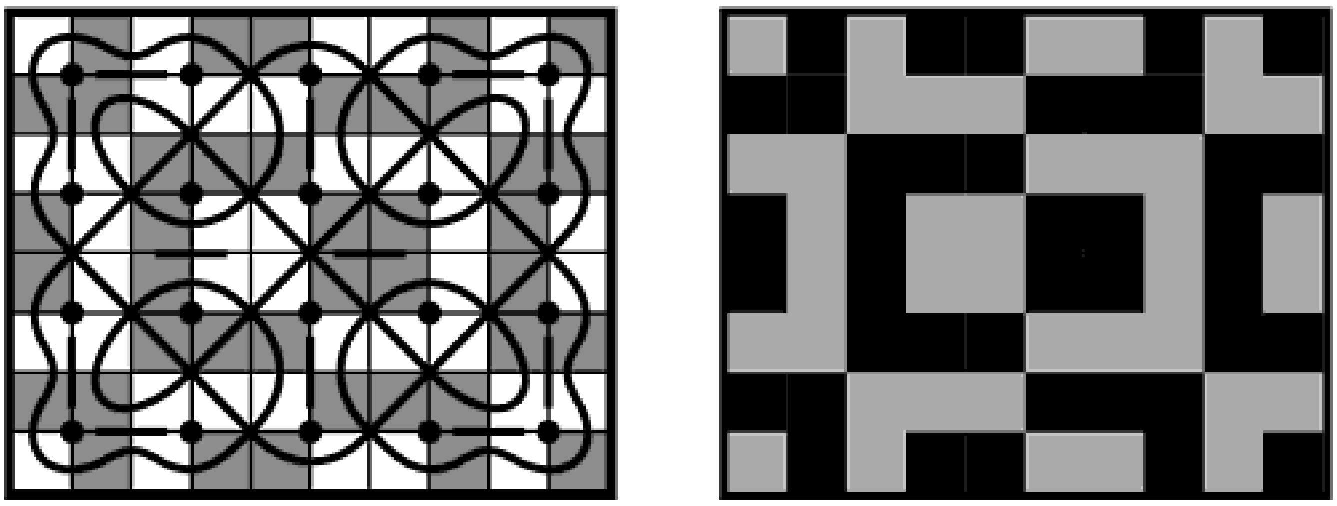
a b c d e

Figure 9.2



a b c

Figure 9.3

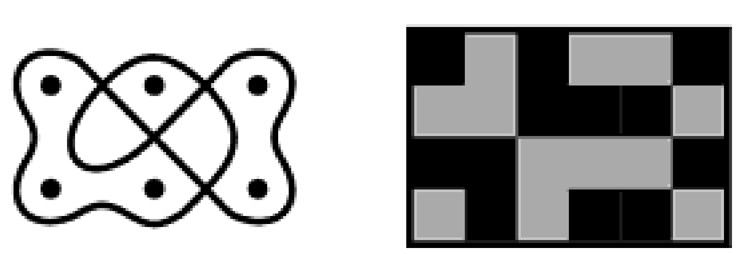


a b

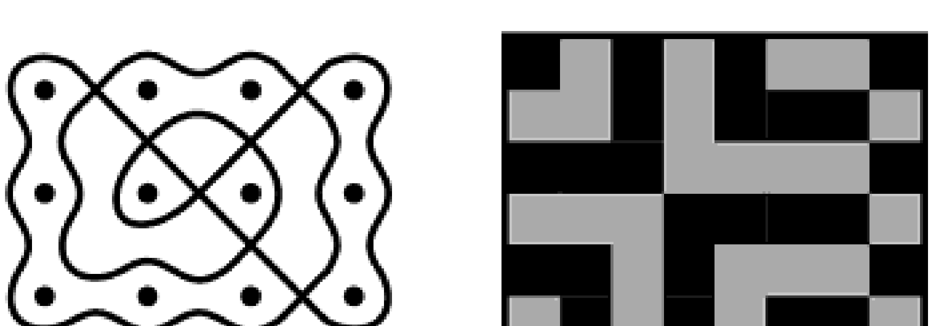
Figure 9.4

This type of black-and-white design I call *Lunda-design*. I discovered them in the context of analyzing a generalization of a type of figure traditionally drawn by Cokwe storytellers in the sand to illustrate their tales, fables and proverbs (cf. Gerdes, 1990, 1995, 1997a). The Cokwe live predominantly in northeastern Angola, a region called Lunda. Hence the name Lunda-design. Lunda-designs have interesting local and global symmetries.

Figure 9.5 presents further examples of Lunda-designs generated by mirror curves that are topologically equivalent to Celtic knots reproduced by Meehan (1991, pp. 123, 122, 142) and Wilson (1983, Pl. 28). Students may be asked to look in the literature for reproductions of Celtic knots, and construct, if possible, their corresponding mirror curves and Lunda-designs. Which properties do all these Lunda-designs have in common? What happens in between neighboring grid points? What happens between the border and the grid points near to it?

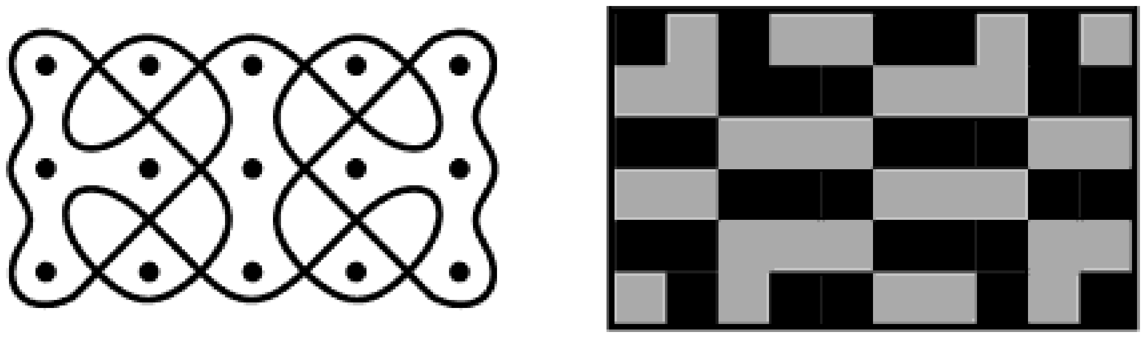


a

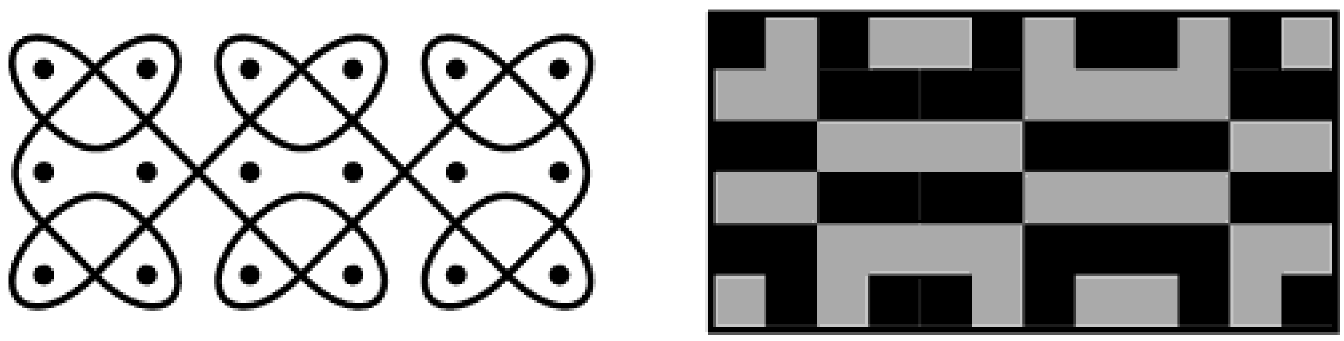


b

Figure 9.5 (first part)



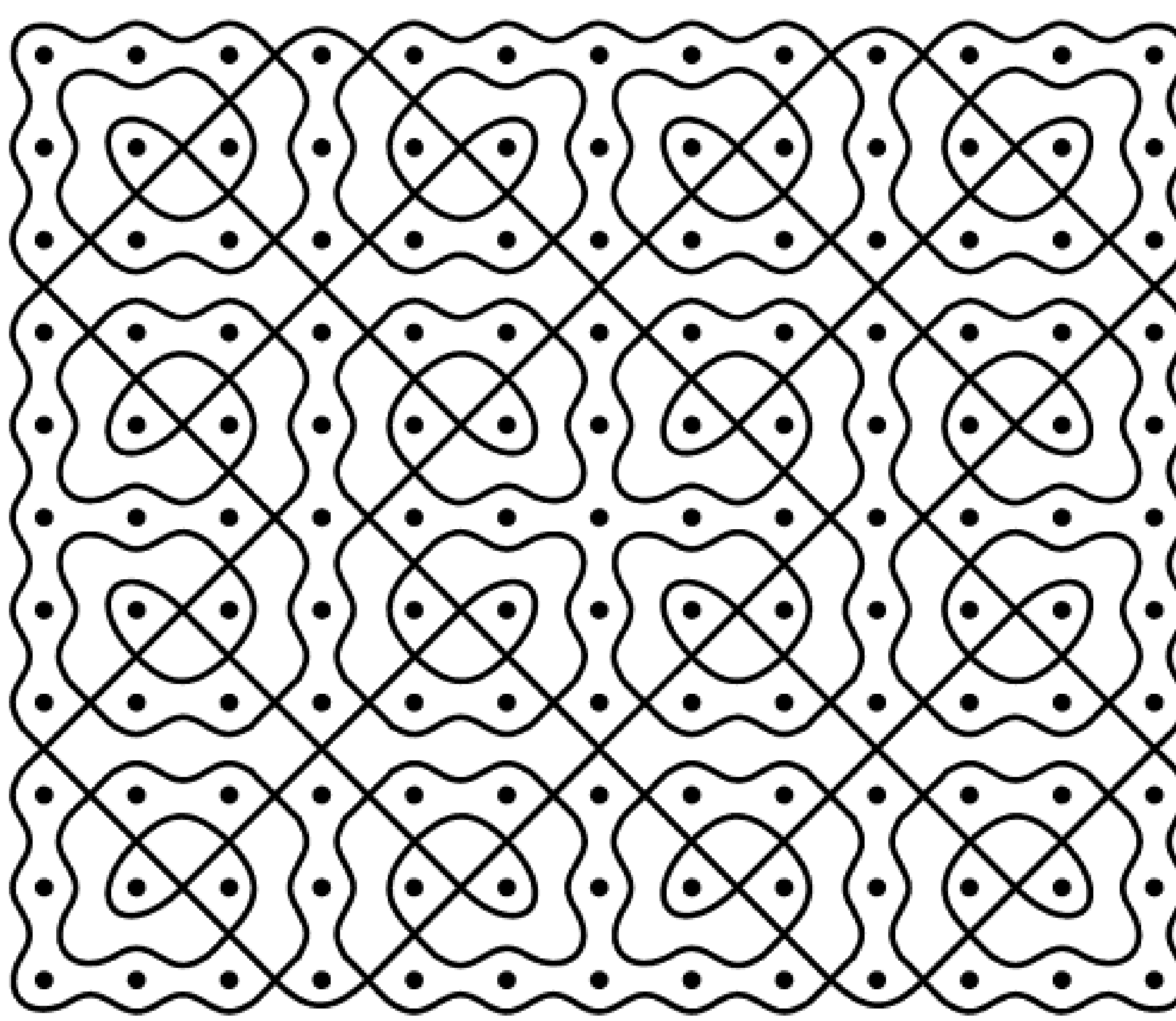
c



d

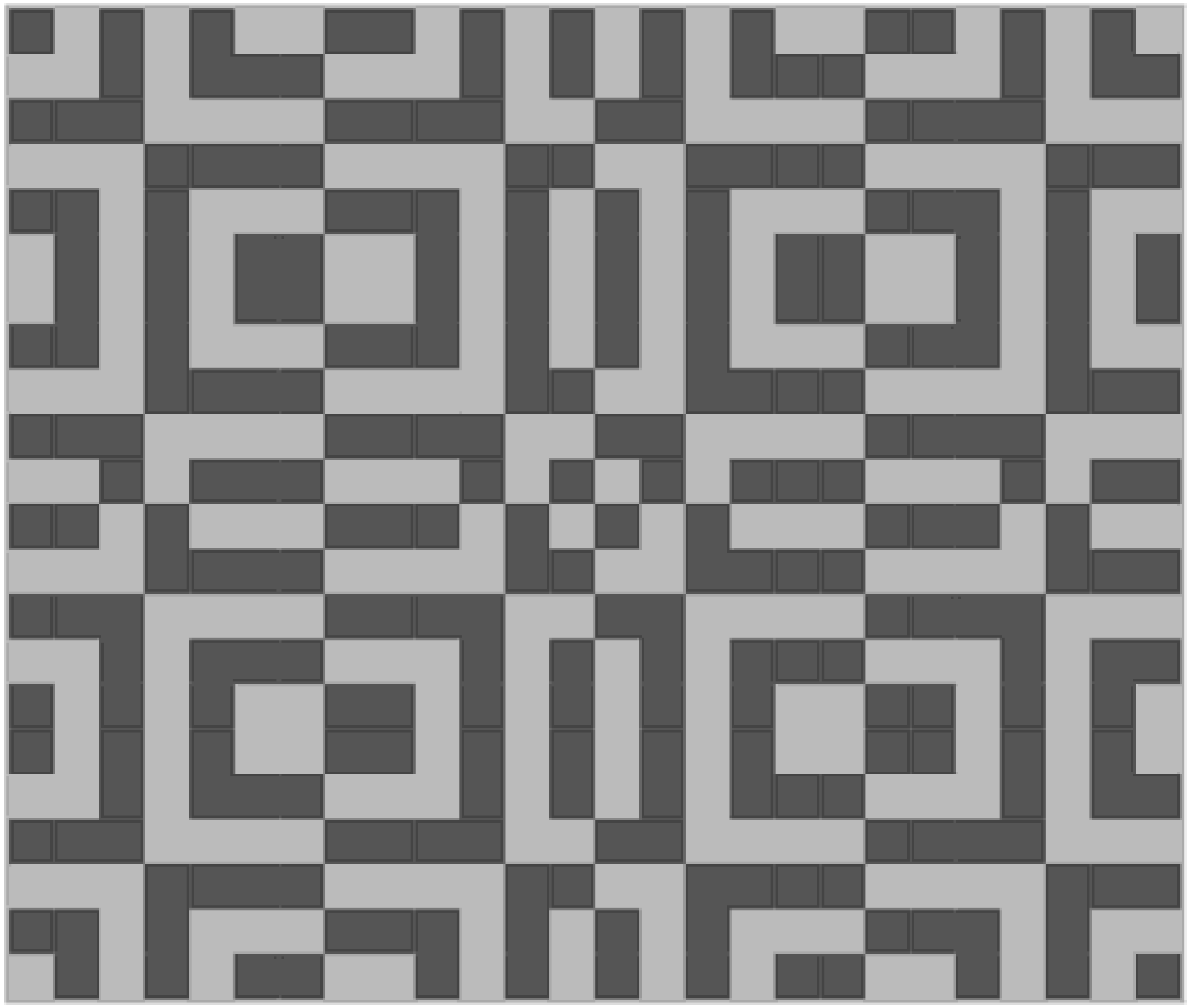
Figure 9.5 (second part)

Once conjectures have been found, they may be tested. For instance, are they verified in the cases of the Lunda-designs presented in Figures 9.6 and 9.7, generated by mirror curves that are topologically equivalent to Celtic knots reproduced by Meehan (1991, p. 130) and Davis (1991, p. 21).

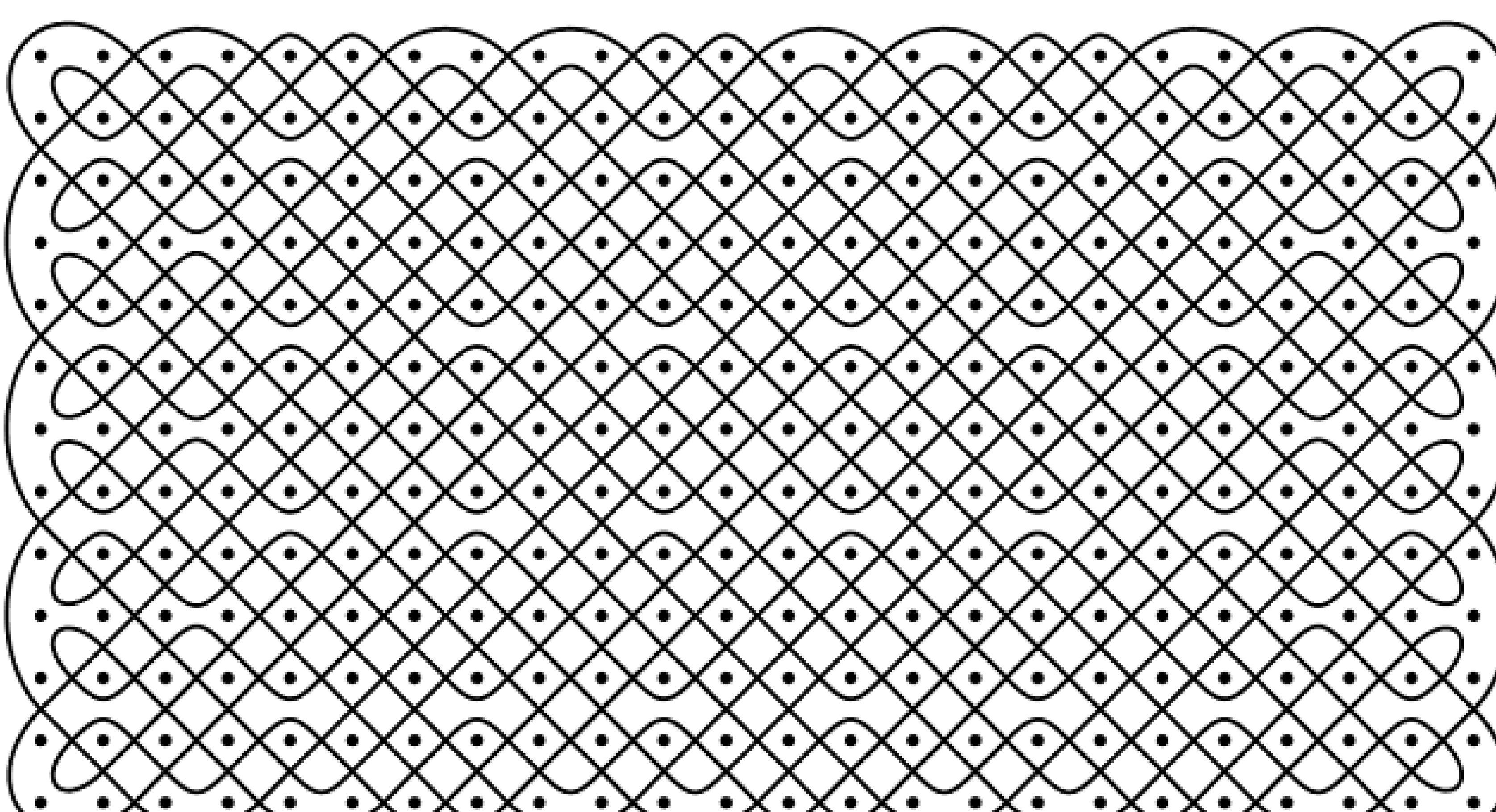


a

Figure 9.6 (first part)

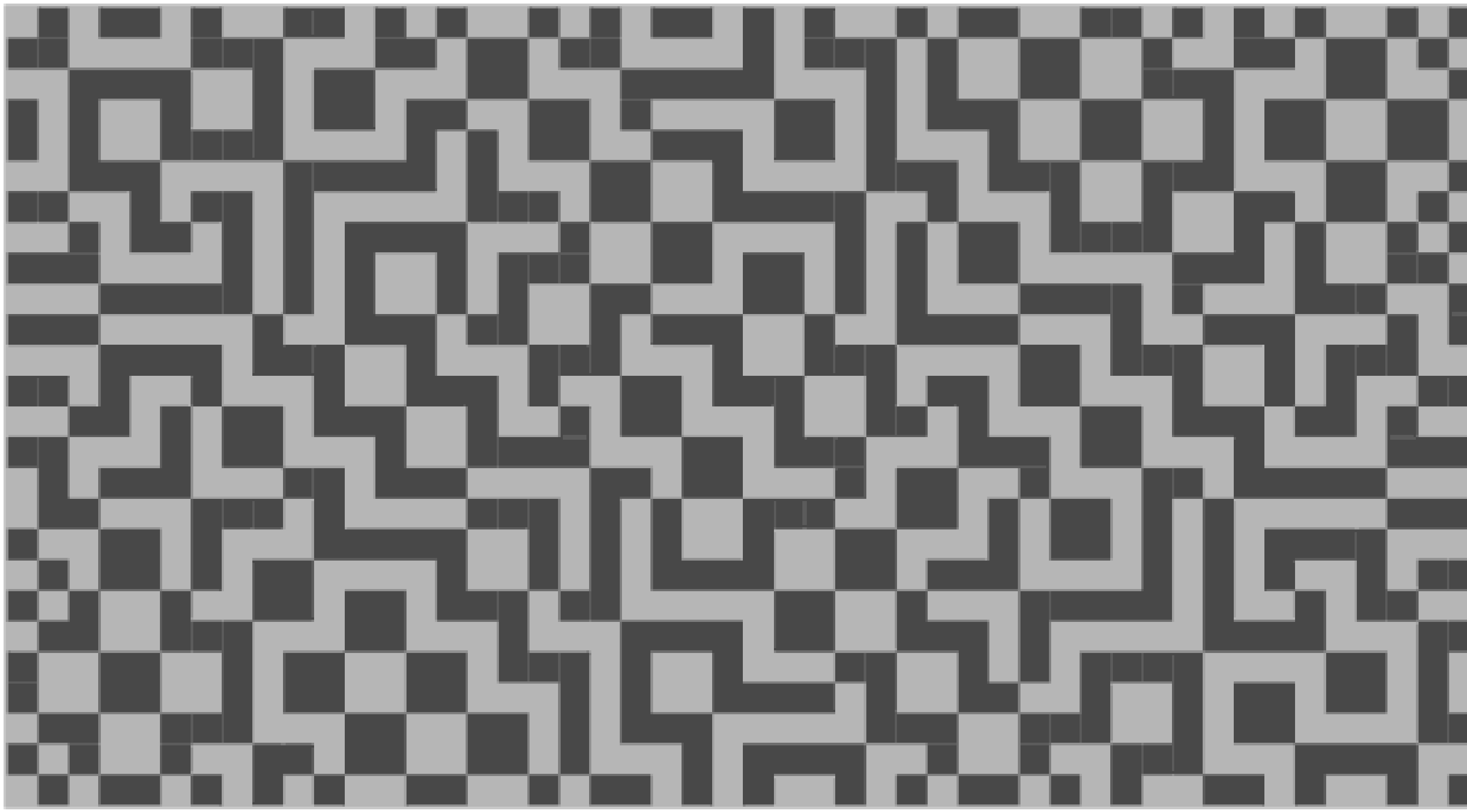


b
Figure 9.6 (second part)



a
Figure 9.7 (first part)

Figure 9.8 displays some symmetrical parts of the Lunda-design in Figure 9.7.



b

Figure 9.7 (second part)

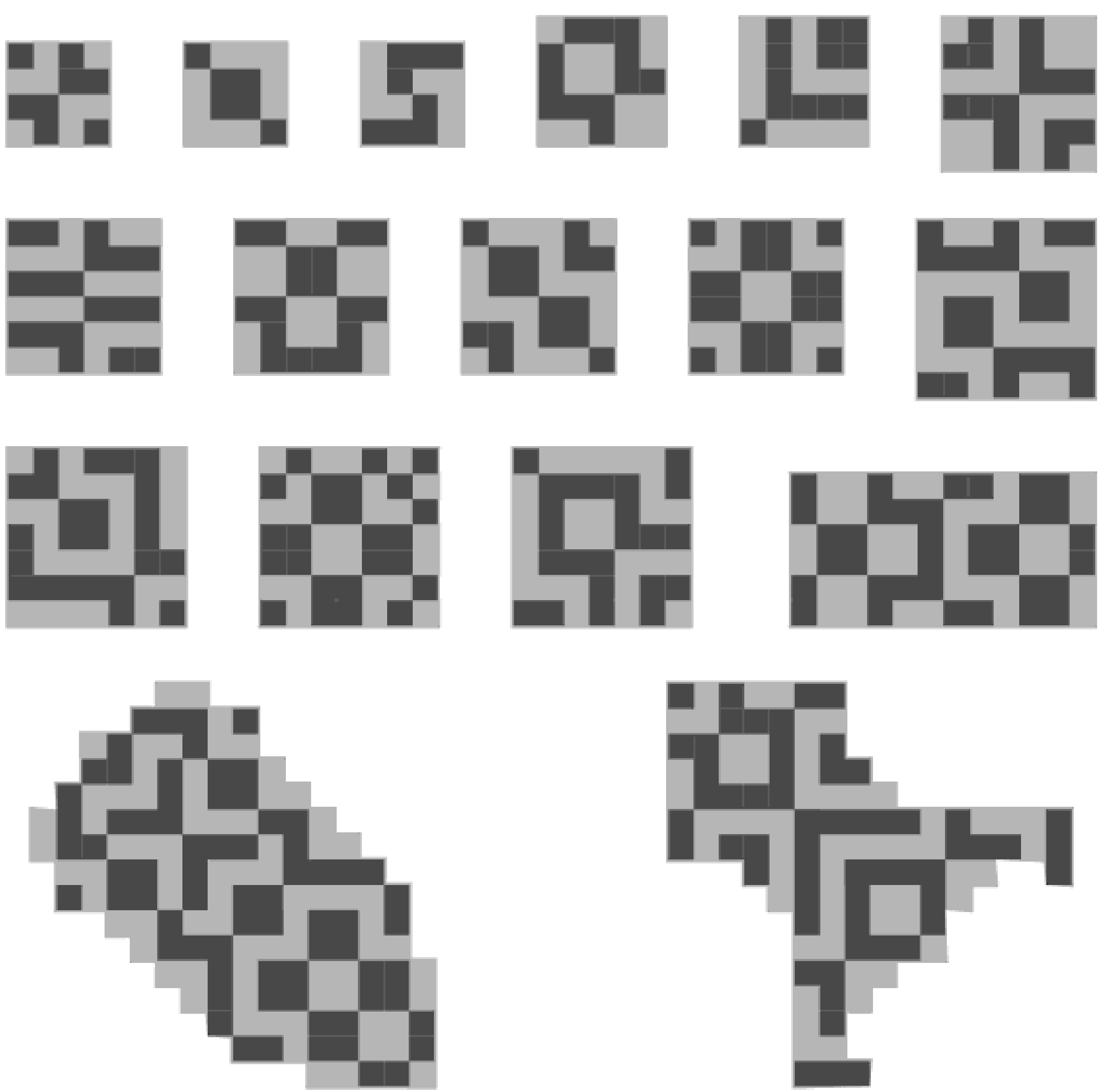


Figure 9.8 (first part)

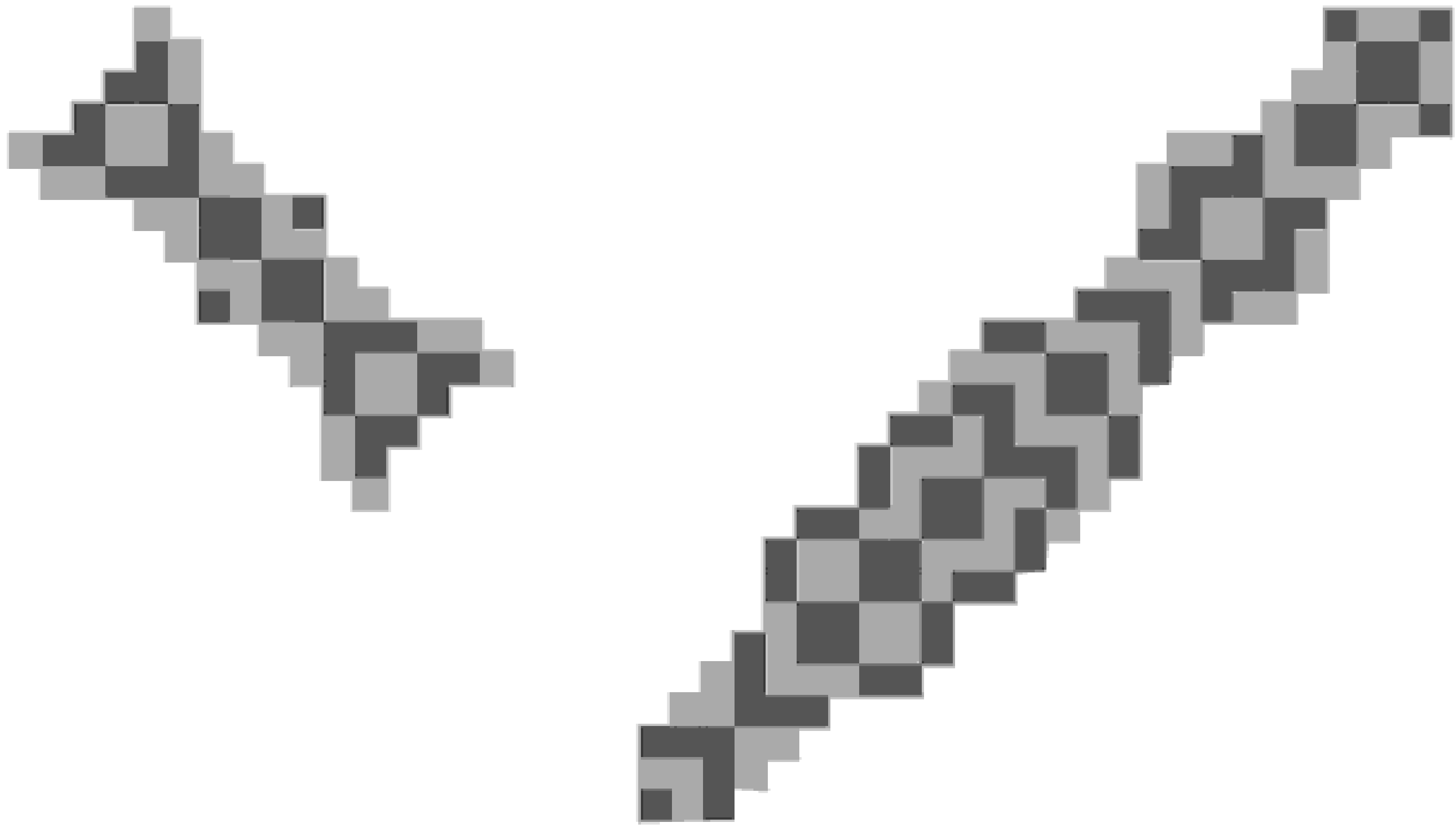
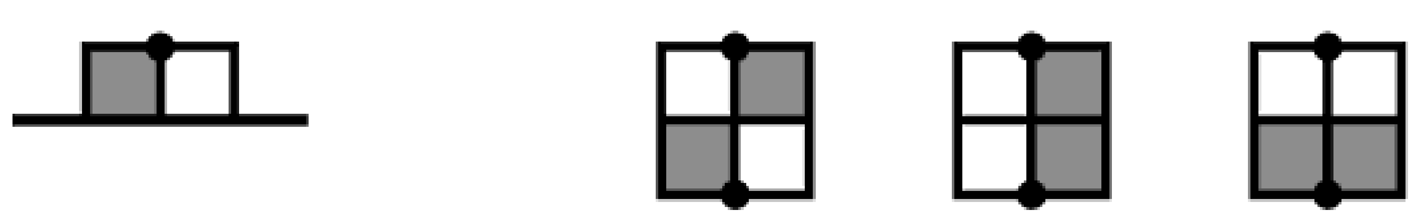


Figure 9.8 (second part)

As may be proven (Gerdes, 1996), Lunda-designs have the following two local (two-color) symmetry properties:

(i) Along the border each grid point always has one black unit square and one white unit square associated with it (see the example in Figure 9.9a);
(ii) Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are black and two are white (see the examples in Figure 9.9b).



a b

Figure 9.9

From this, it follows that Lunda-designs have a global symmetry, characterized by the phenomenon that:

(iii) in each row (and in each column) there are as many black unit squares as there are white unit squares.

Conversely, as for each black-and-white design that satisfies the characteristics (i) and (ii) a mirror curve may be produced that generates it (for a proof, see Appendix 1), these characteristics can be used to define Lunda-designs.

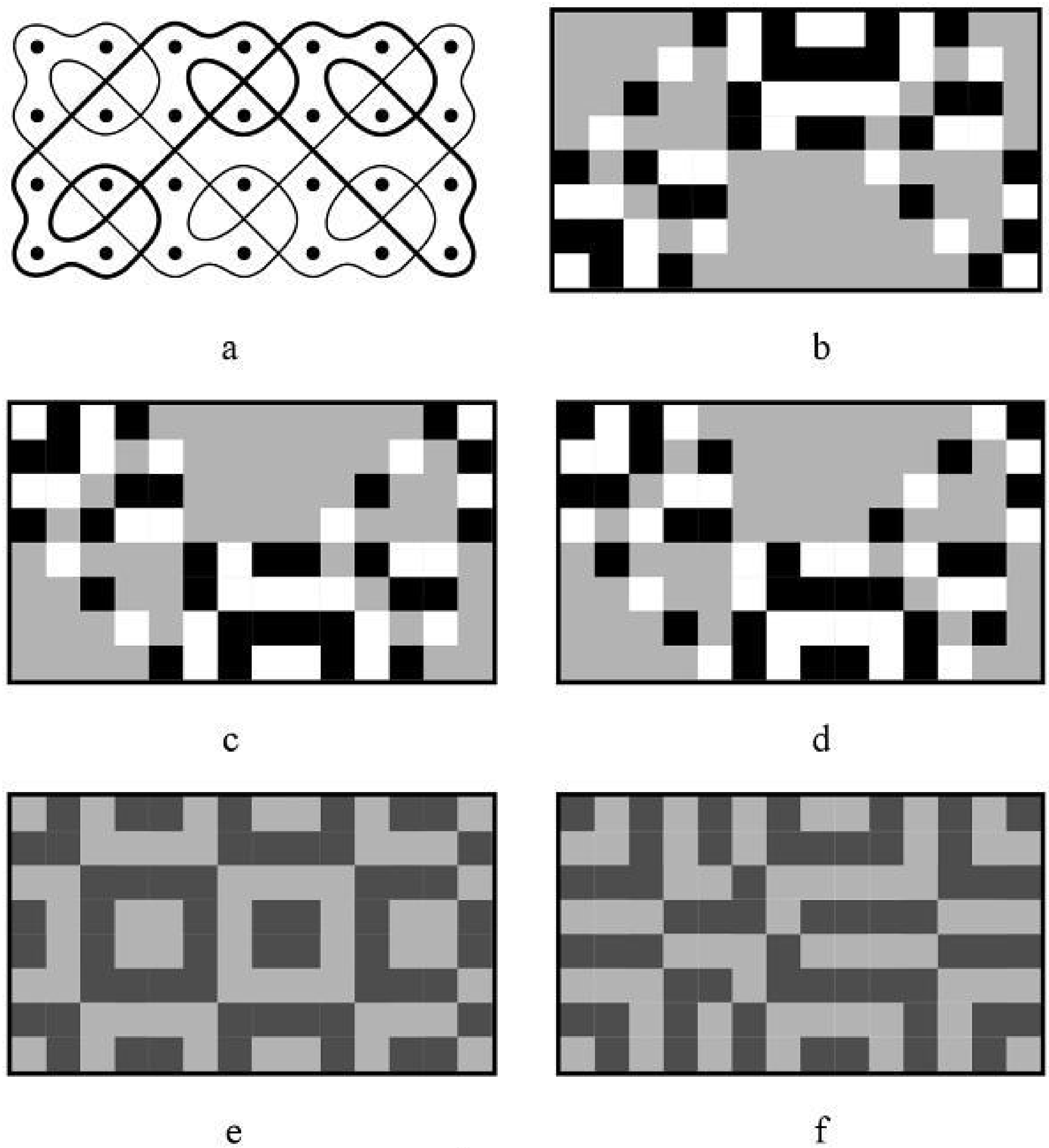


Figure 9.10

In each of the examples of Celtic knots presented so far, the mirror curve passes through *all* unit squares of the rectangular grid. In other words, the knots are composed of only one line. Let us call such knots *monolinear* or 1-linear. However, there are polylinear Celtic knots, composed out of more than one line. Figure 9.10a presents the topological equivalent of a 2-linear knot reproduced by Meehan (1991, p. 146). If we color now alternately black and white the unit squares through which that curve passes that 'starts' in the lower left corner of the grid, only part (in this case, half) of the unit squares will be colored (see Figure 9.10b). As there exist two possibilities to color the unit squares through which the second closed curve passes (see Figure 9.10c and d), there emerge two associated black-and-white designs (see Figure 9.10e and f). It is easy to verify that both are Lunda-designs. More in general, it may be shown that a n-linear knot, topologically equivalent to a n-linear mirror curve design, generates 2^{n-1} Lunda-designs. Both associated Lunda-designs in Figure 9.10e and f have a two-color symmetry axis: reflection in their horizontal axes interchanges black and white.

All Lunda-designs constructed in this paper have two-color symmetries. The Lunda-designs in Figure 9.5a and b have horizontal two-color axes, whereas their generating mirror curves are not symmetrical. The Lunda-designs in Figures 9.3, 9.4, 9.5c and d, and 9.6 have horizontal and vertical two-color symmetries. The Lunda-design in Figure 9.7 has a two-color rotational symmetry: a half turn about its centre interchanges black and white. This attractive Lunda-design displays various other interesting local symmetries, as the reader may verify.

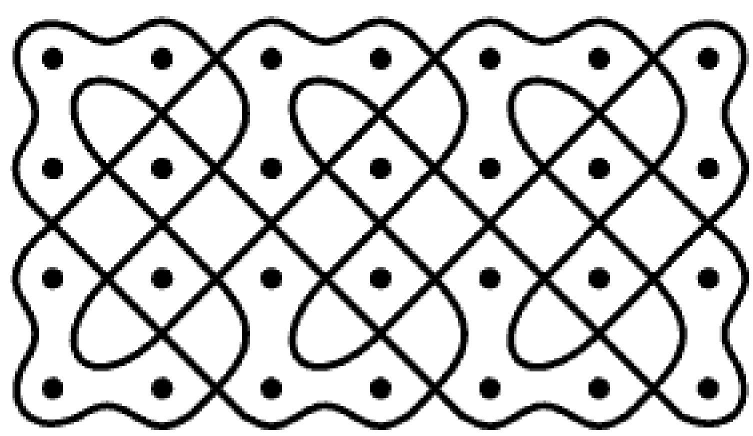


Figure 9.11

Figure 9.11 presents the topological equivalent of a monolinear Celtic knot. The Lunda-design generated by this mirror curve is the one already presented in Figure 9.10e. This constitutes a concrete example of the fact that distinct knots may generate the same Lunda-design. The general question of how the number of Lunda-designs depends on the dimensions of the reference grids is still open. Some answers for particular classes of Lunda-designs have been found (cf. Gerdes, 1996).

Another interesting topic for further investigation is that of sequences of Lunda-designs. Figure 9.12 presents the first elements of a sequence of mirror curves. The fifth element is topologically equivalent to a monolinear Celtic knot (see Figure 9.13), reproduced by Jones (1856, T. LXIV, no. 10). This sequence of mirror curves generates a sequence of Lunda-designs, of which the first elements are represented in Figure 9.14. Is it possible to predict how this sequence will continue without constructing first the mirror curves and then generating the corresponding Lunda-designs?

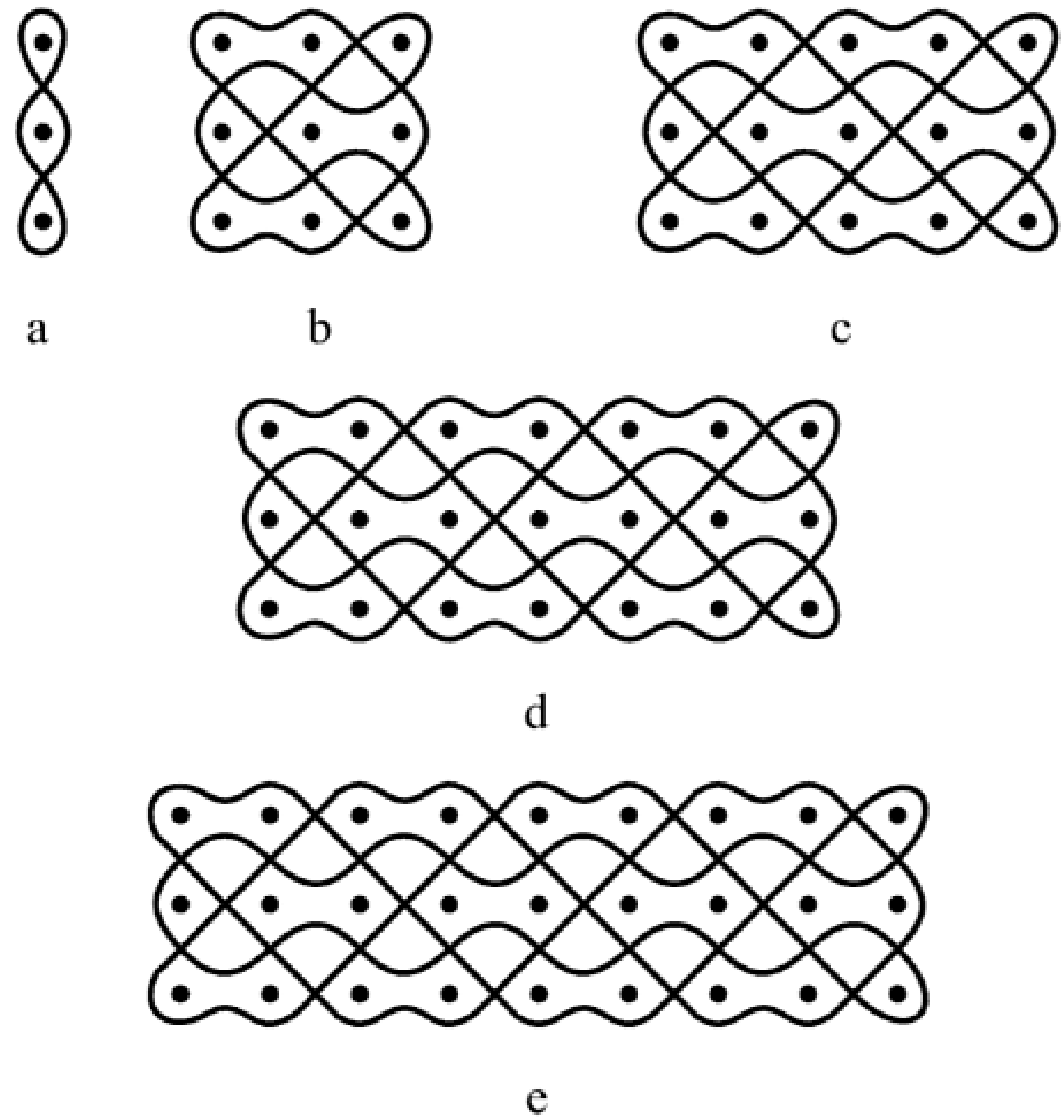


Figure 9.12

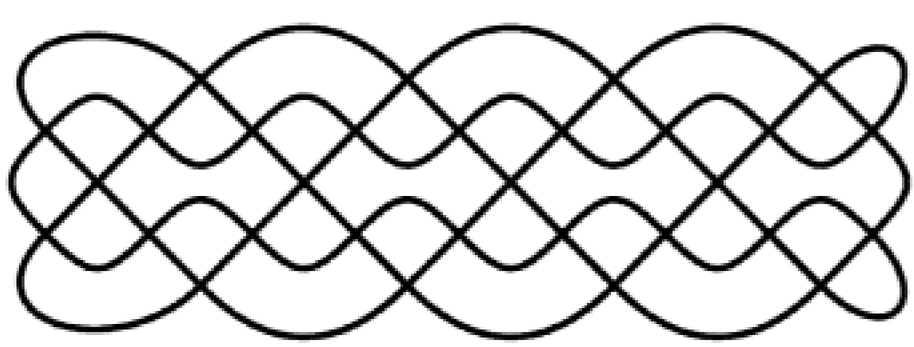


Figure 9.13

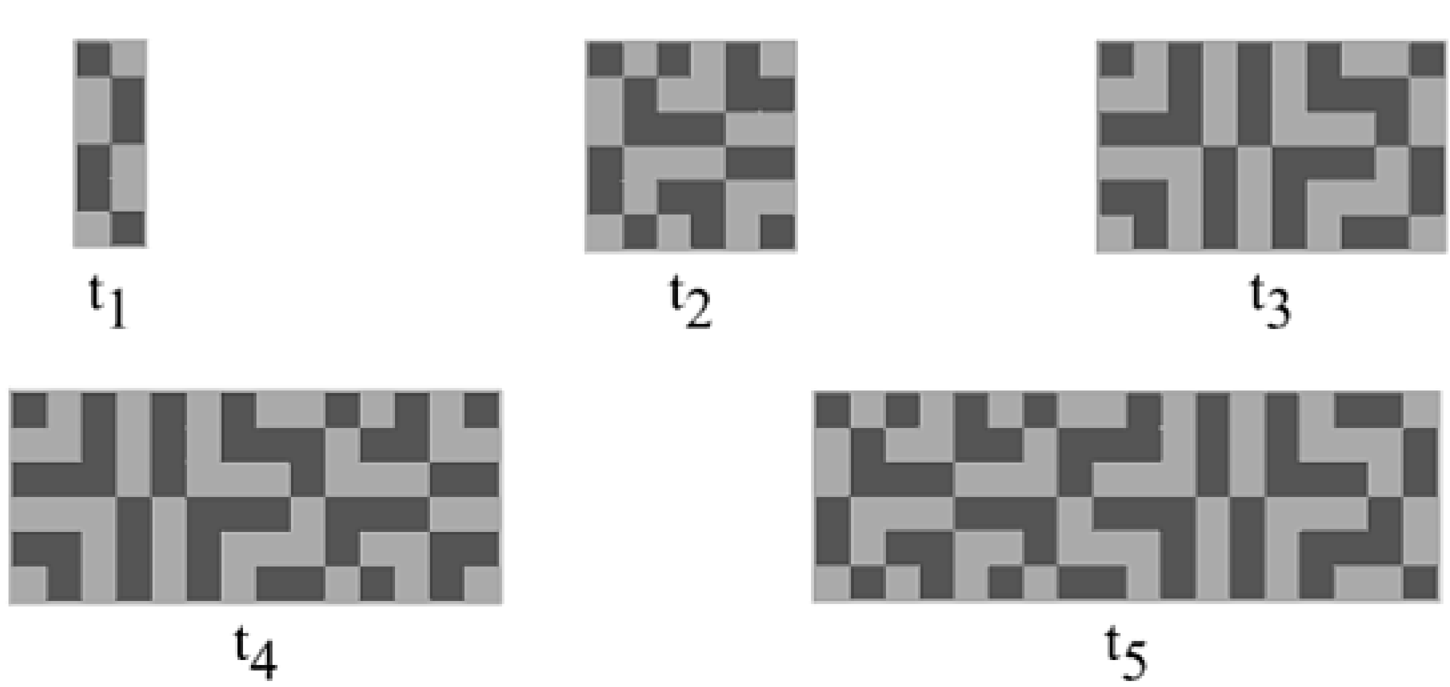
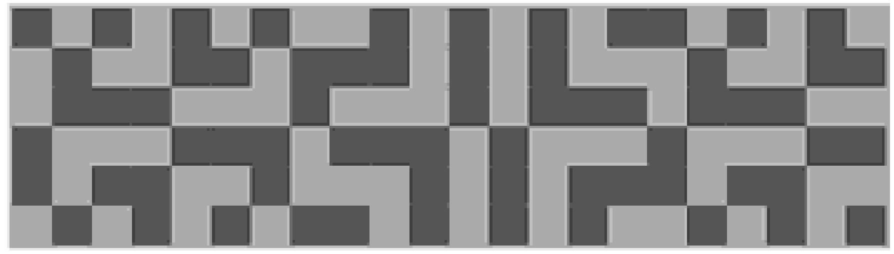
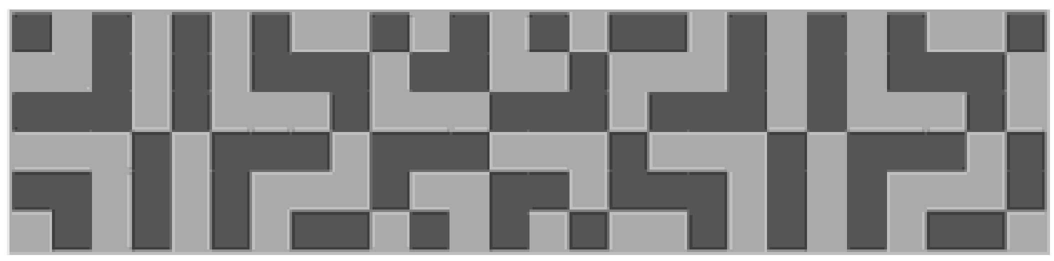


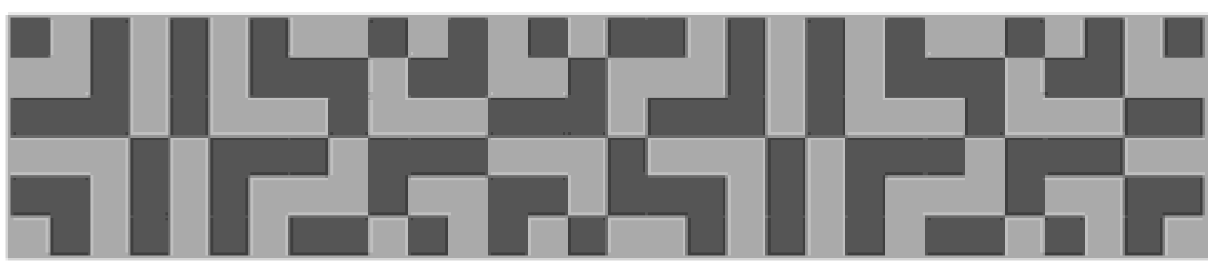
Figure 9.14 (first part)



t_6



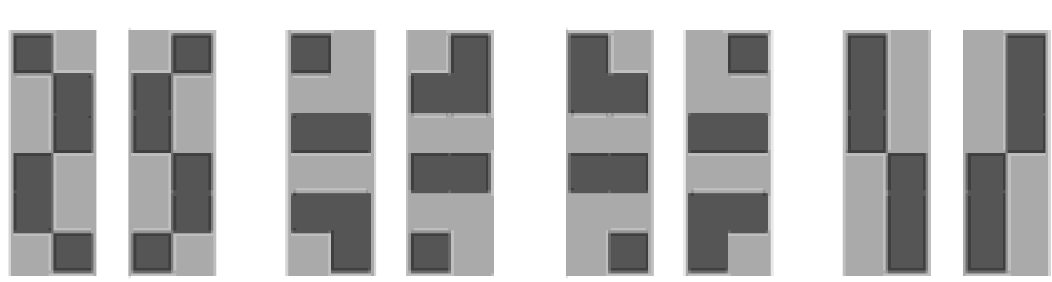
t_7



t_8

Figure 9.14 (second part)

One way to find an answer to this question is the following. Let us divide each of the Lunda-designs belonging to the sequence into vertical slices of a width of to unit squares. There are 8 distinct slice types, as illustrated in Figure 9.15: A and A', B and B', C and C', and D and D' are each other negative. When we rotate C about its centre through an angle of 180^{o} we obtain B. Using this notation we have t_1 = A, t_2 = ABC, t_3 = BDDCA', etc. (see Figure 9.16). Can we discover some structure in this letter pattern?



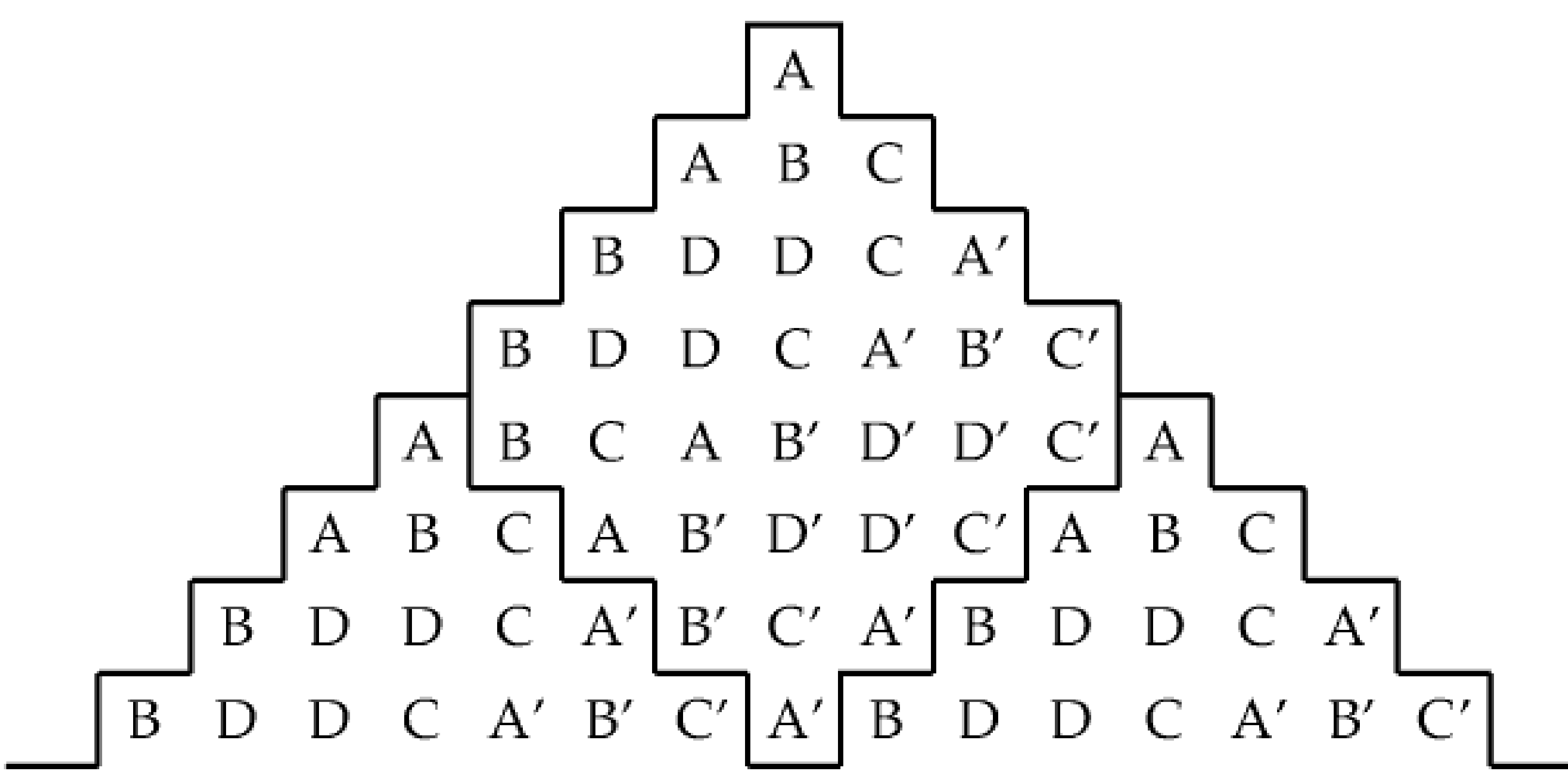
A A′ B B′ C C′ D D′

Figure 9.15

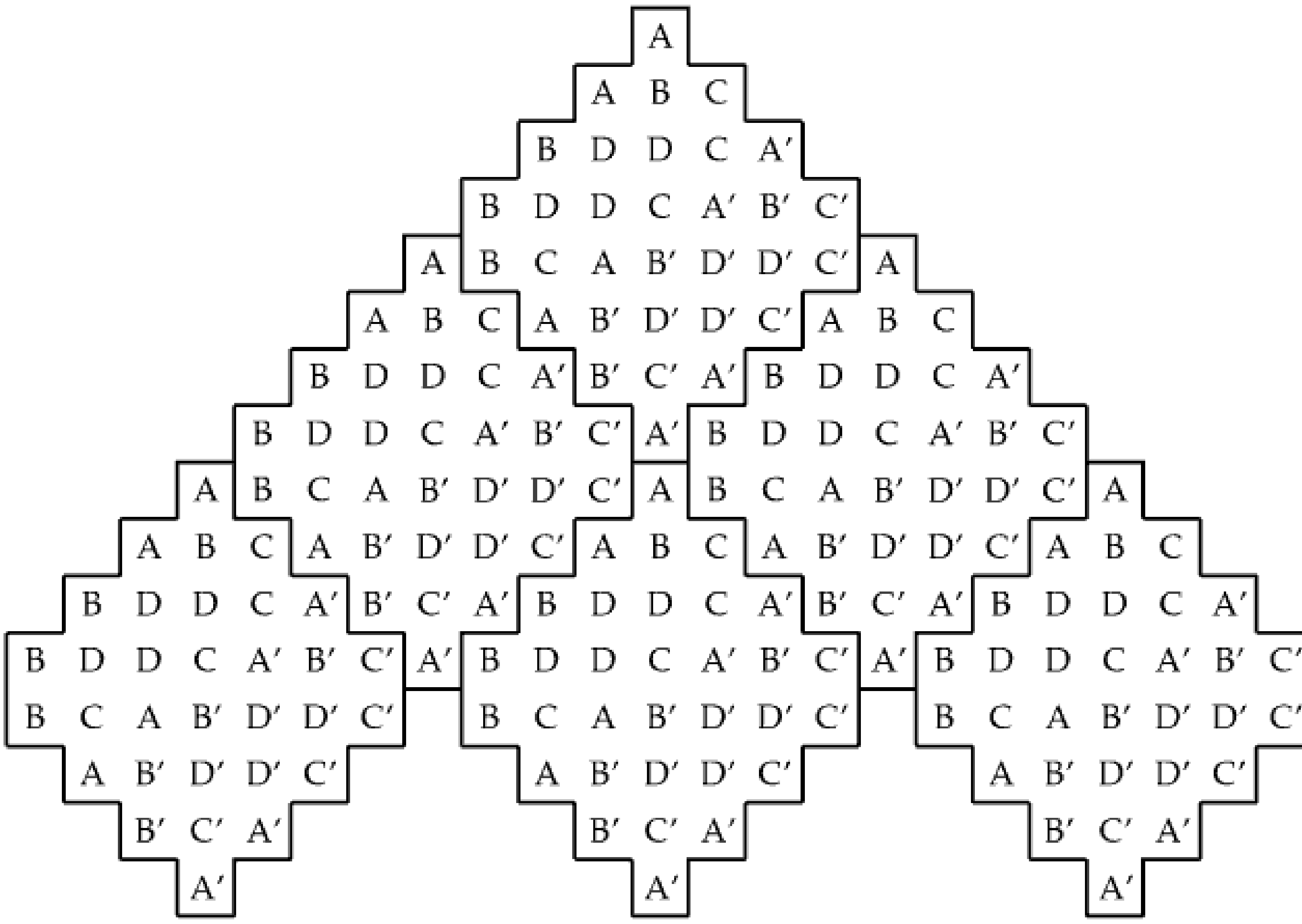
1								A							
2							A	B	C						
3						B	D	D	C	A′					
4					B	D	D	C	A′	B′	C′				
5				A	B	C	A	B′	D′	D′	C′	A			
6			A	B	C	A	B′	D′	D′	C′	A	B	C		
7		B	D	D	C	A′	B′	C′	A′	B	D	D	C	A′	
8	B	D	D	C	A′	B′	C′	A′	B	D	D	C	A′	B′	C′

Figure 9.16

In each of the diagonal directions, there seems to be cycle of length 4: (AABB), (BDDB),..., (ACA'C'), (BCB'C'), etc. (see Figure 9.17a). Extrapolation on the basis of these experimental data leads us to conjecture a letter pattern built up out of repeating 'zigzag rhombi' (see Figure 9.17b).



a
Figure 9.17



b
Figure 9.17

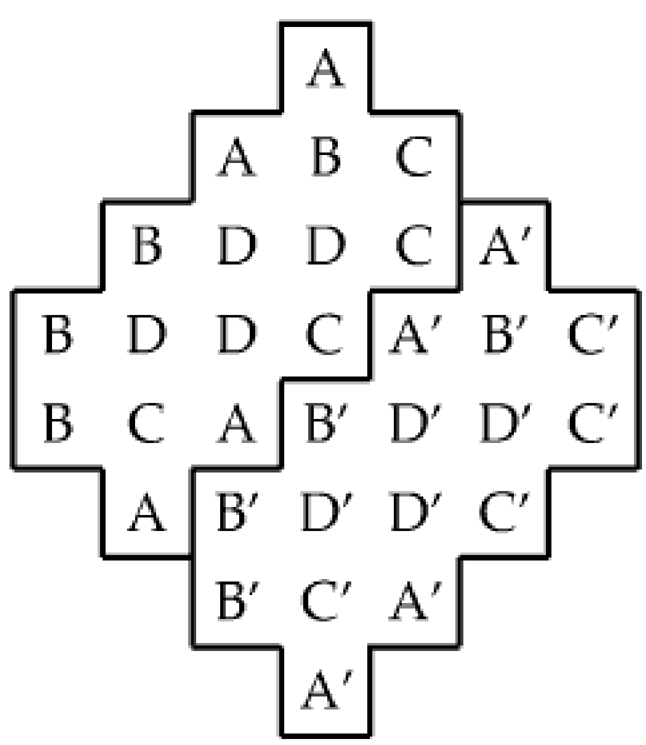
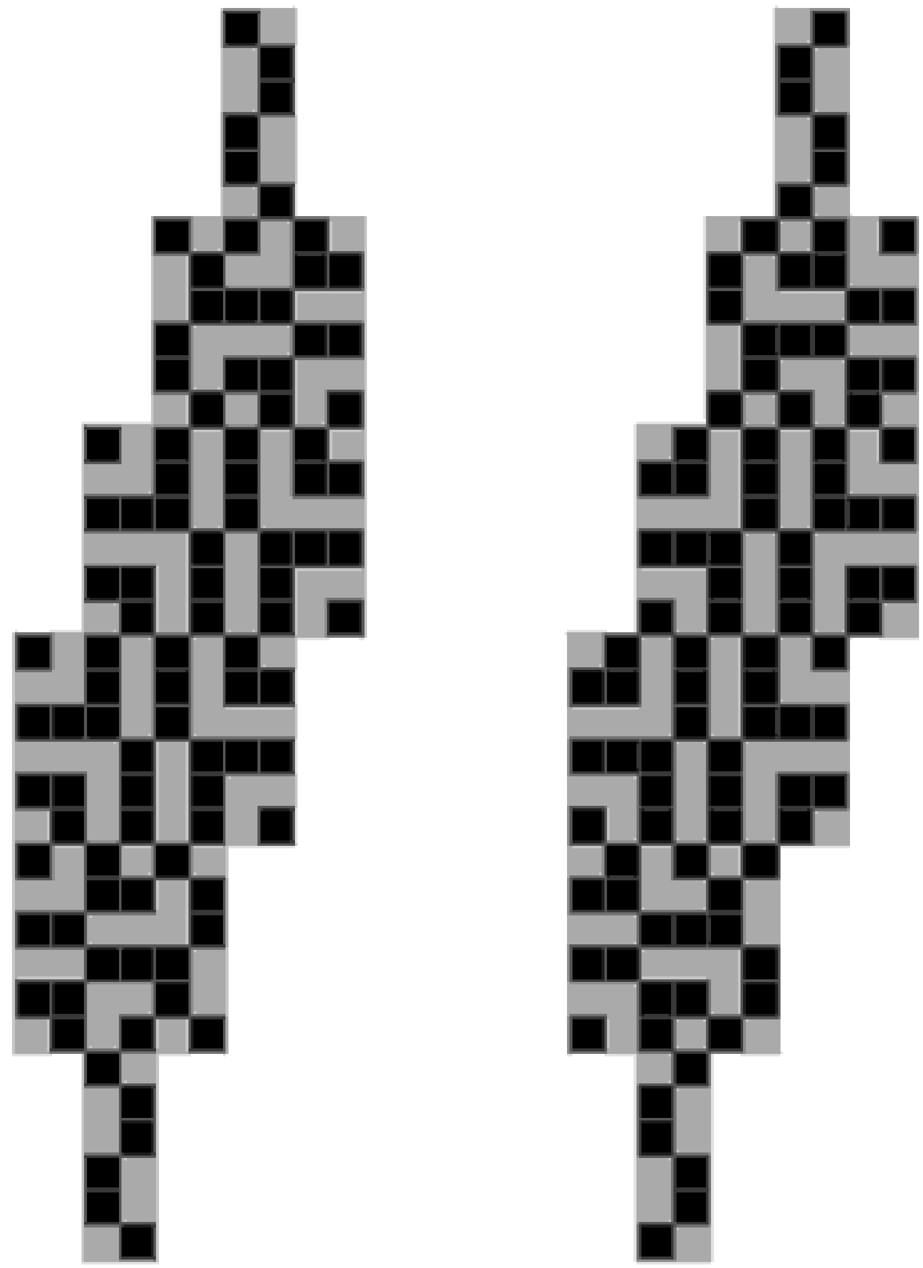


Figure 9.18

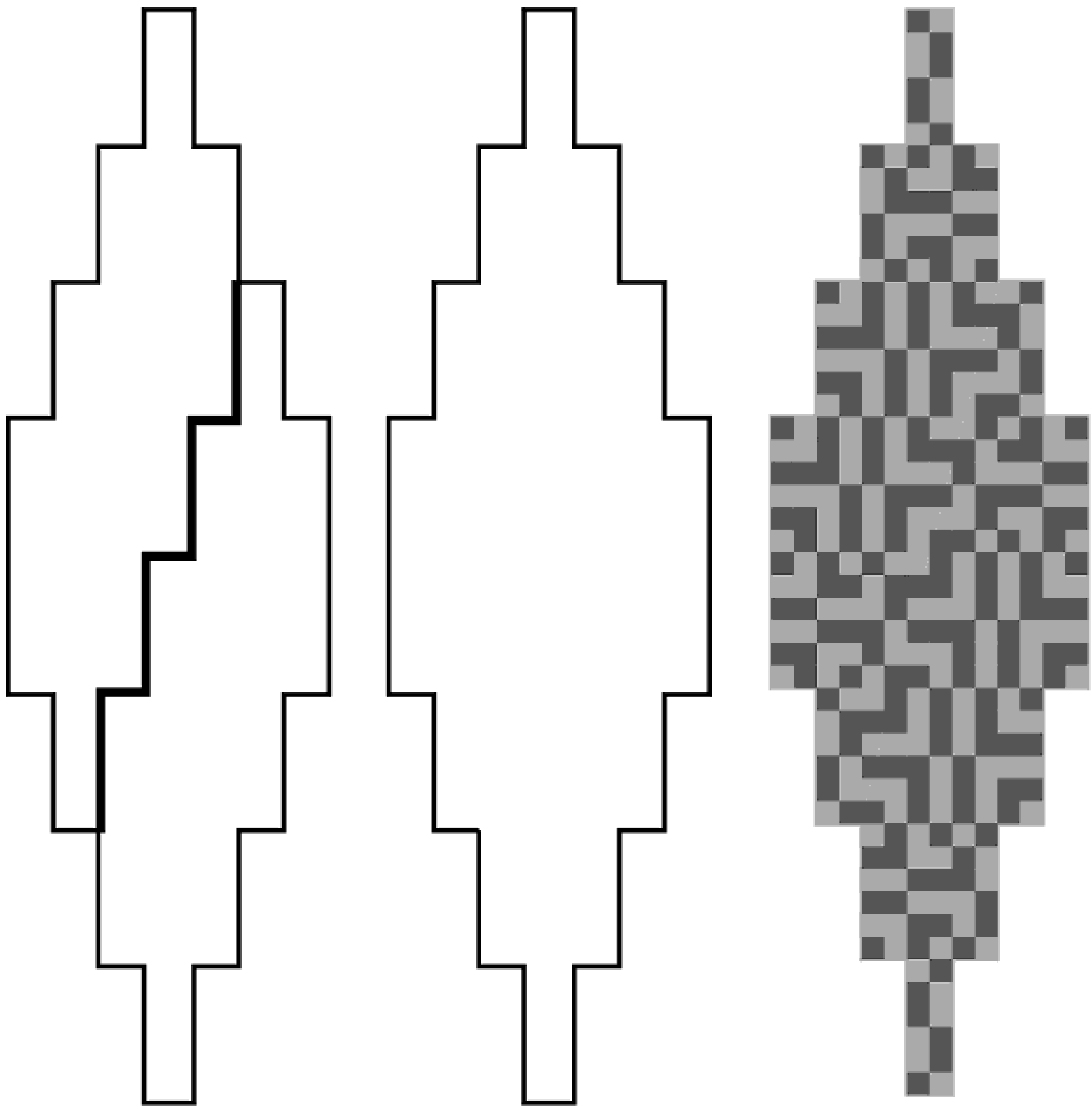
The ‘zigzag rhombus’ has interesting symmetries. It is composed of two halves (see Figure 9.18): each of the letter elements of the second one is the negative of the corresponding letter element of the first one. Moreover each half is invariant under a half turn about its centre (see Figure 9.19).



a b

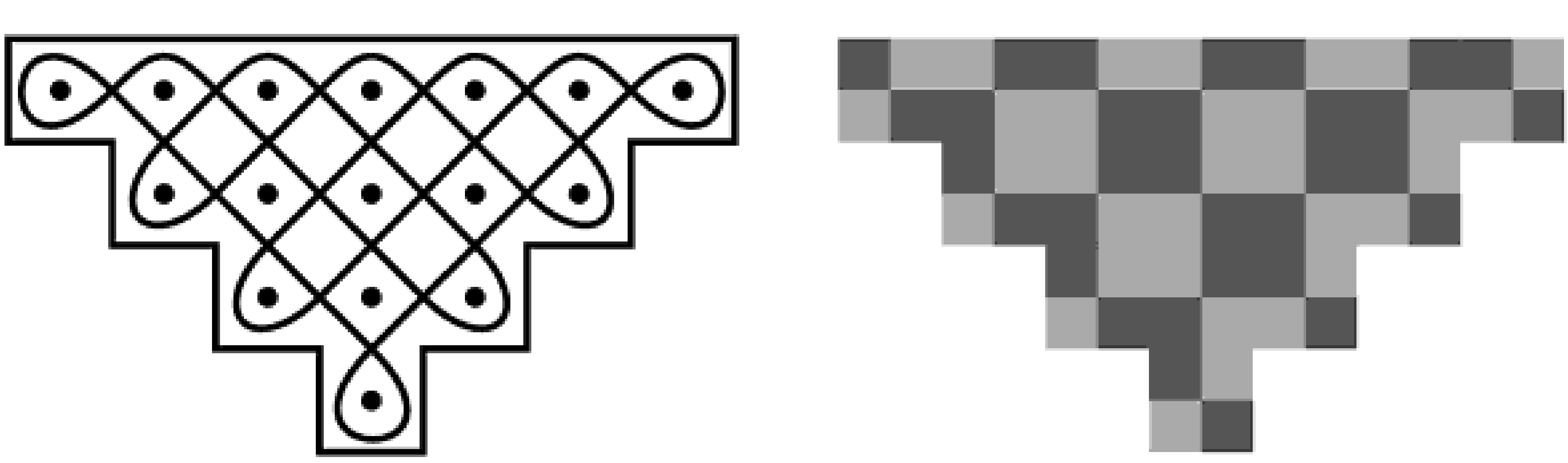
Figure 9.19

Figure 9.20 presents the black-and-white 'zigzag rhombus'. We may call the black-and-white designs in Figure 9.19 and 9.20 *polyominal Lunda-designs* (cf. Gerdes, 1996, 1997). Figure 9.21b presents another polyominal Lunda-design, generated by a Celtic knot (Figure 9.21a), reproduced by Jones (1856, Pl. LXIV). For other generalizations, like circular, hexagonal, and polyhedral Lunda-designs and (multicolor) Lunda-k-designs, and Lunda-strip and plane patterns, see Gerdes (1996, 1997, 1998a). Figure 2.27 shows the first three phases of building up a Lunda-fractal, generated by the Celtic foundation knot (cf. Figure 9.1a and 9.3c).



a b c

Figure 9.20



a b

Figure 9.21

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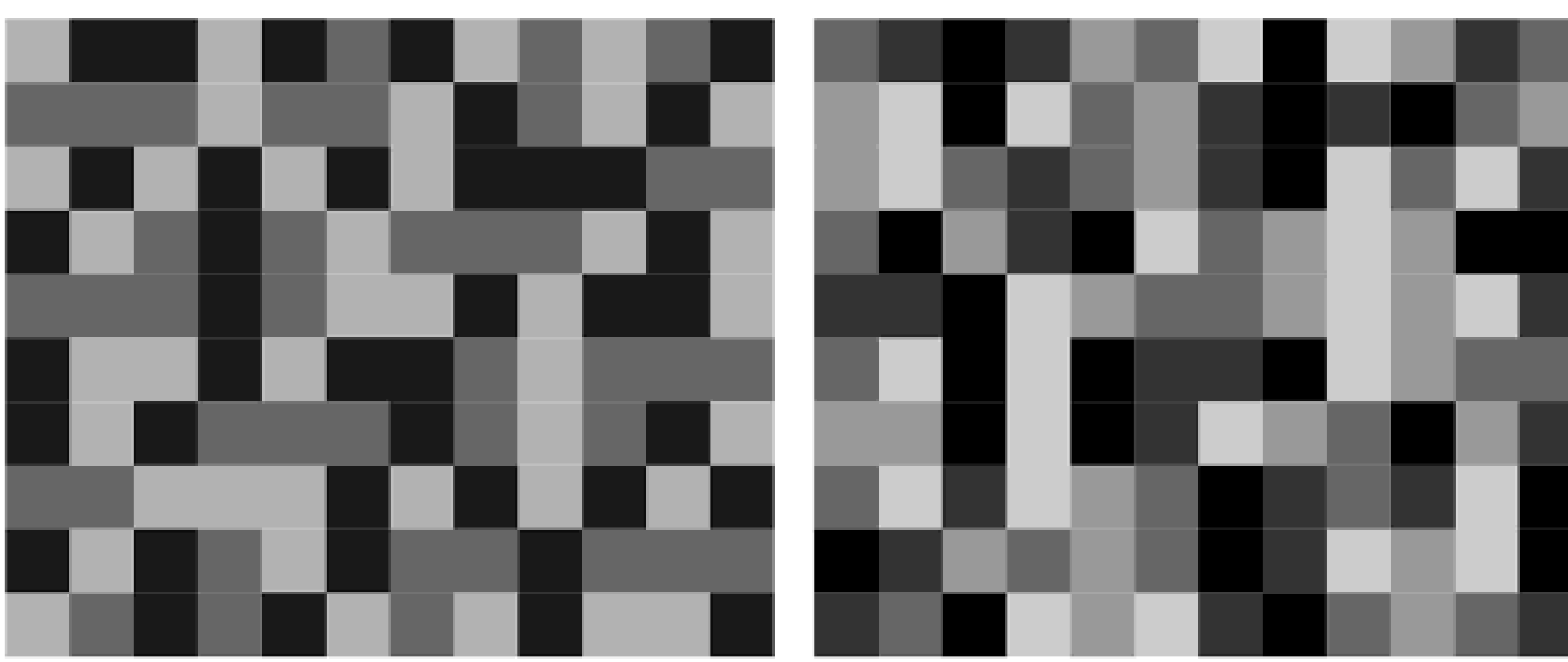
Chapter 10

RECENT VARIATIONS AND GENERALIZATIONS [1]

Instead of enumerating the unit squares through which a mirror-curve passes modulo 2 and thus producing a Lunda-design, it is possible to enumerate them modulo *s*, where *s* designates any divisor of the total number of unit squares. For instance, Figure 10.1 displays the mod 3 and mod 5 designs generated in this way by the Cokwe chased-chicken mirror-curve (Figure 1.1). The question arises how to characterize the local and global symmetries of these mirror-curve-modulo-s designs. Conversely, is it possible to define these designs on the basis of their symmetries?

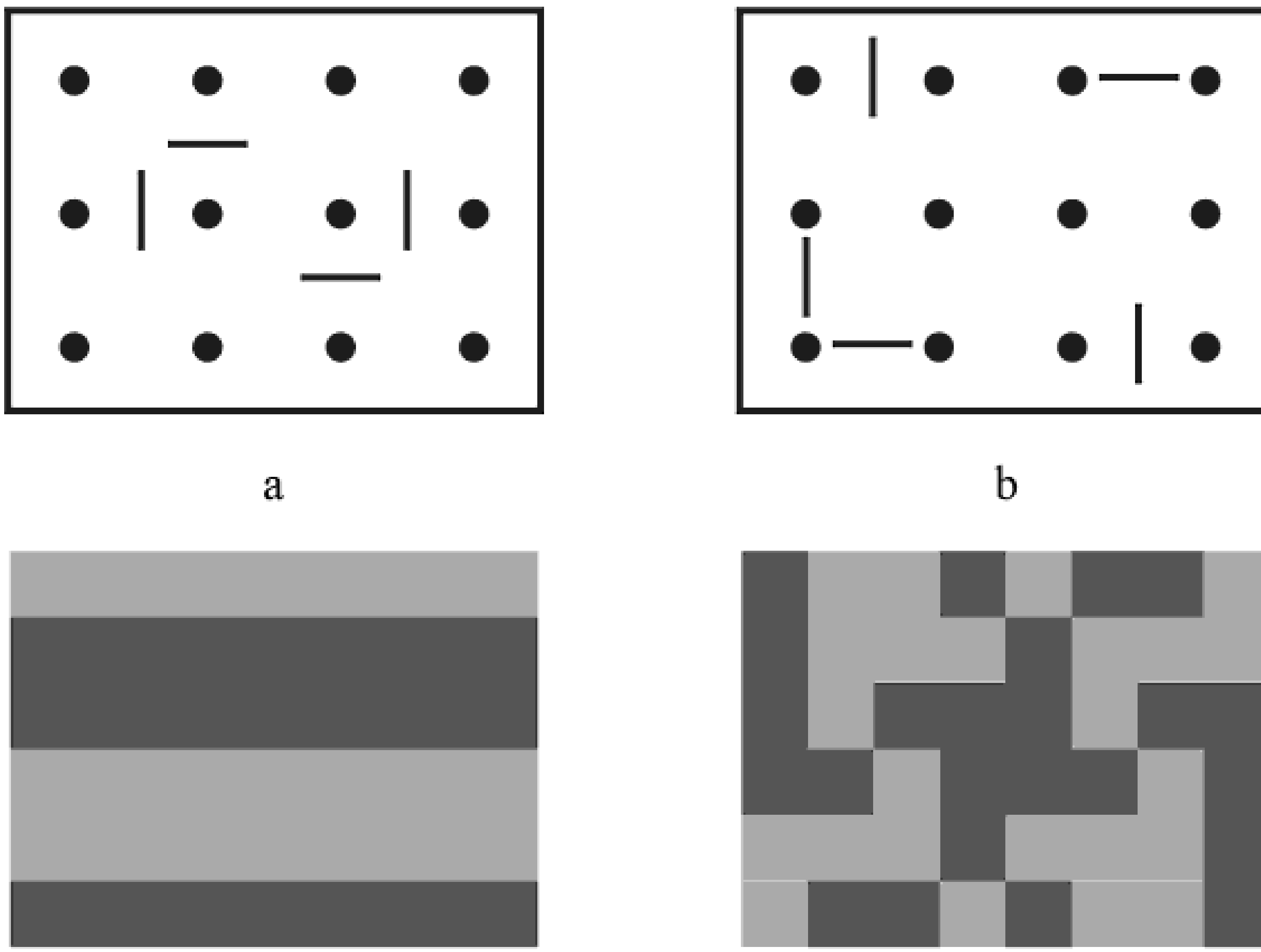


a b

Figure 10.1

1 Last section of the paper "Symmetrical explorations inspired by the study of African cultural activities," published in: István Hargittai & Torvand Laurent (Eds.), *Symmetry 2000*, Portland Press, London, 2002, 75-89.

Instead of enumerating the unit squares mod 2, that is, 0101, etc. one may enumerate them 0011, etc. If this is done, regular mirror-curves generate horizontal or vertical bar designs. Figure 10.2b illustrate the example generated by the mirror-design in Figure 10.2a. Figure 10.2c displays an irregular mirror-design and its corresponding, symmetrical **0011-design** (Figure 10.2d).

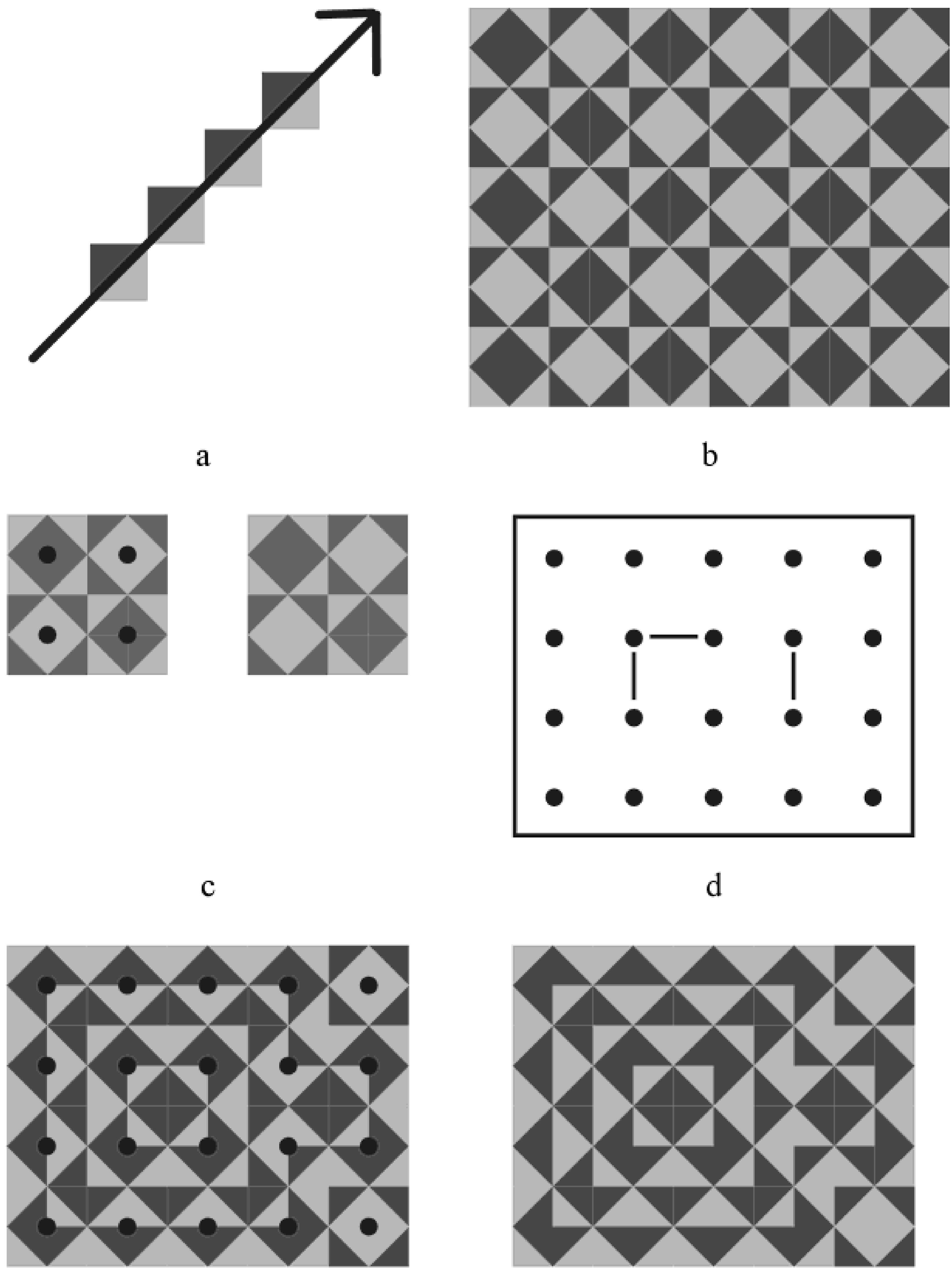


a b

c d

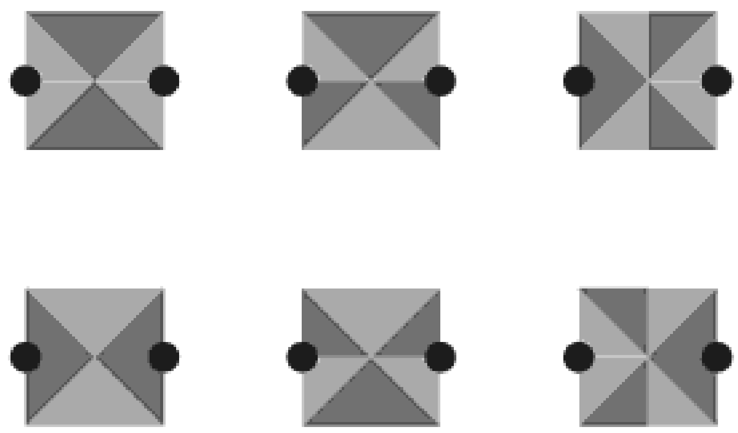
Figure 10.2

Instead of enumerating the unit squares mod 2 or coloring them alternately black and white, it is possible to color them all in the same way. For instance, it is possible to color each unit square through which the (polygonal) mirror-curve passes on its right side black and on its left side white (light grey in the Figures), as Figure 10.3a indicates. For instance, the chased-chicken mirror-curve produces the design in Figure 10.3b. All regular mirror-curves generate similar designs of repeating blocks (Figure 10.3c). Irregular mirror-designs and their corresponding mirror-curves may produce other types of **right-flag-designs**. For example, the mirror-design in Figure 10.3d generates the right-flag-design in Figure 10.3e. The local symmetries of these right-flag-designs are characterized by three pairs (positive and negative) of possible situations between horizontal neighboring grid points (Figure 10.3f) and one pair of possible situations between a border grid point and the border (Figure 10.3g).



e

Figure 10.3 (first part)

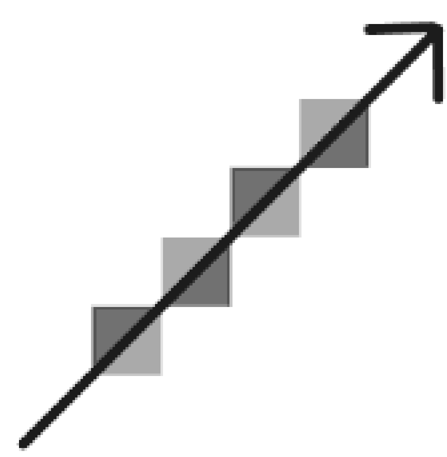


f

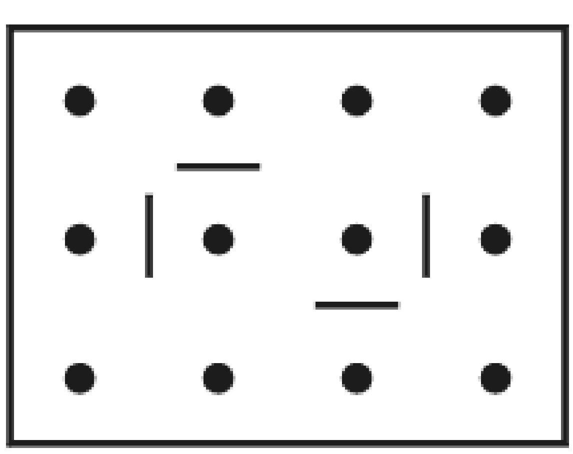
g

Figure 10.3 (second part)

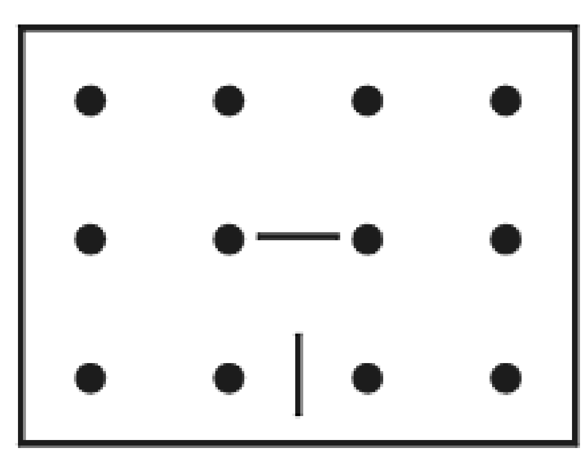
What will happen if the flags alternate from left to right (Figure 10.4a) instead of being always on the right side? Figures 10.4b and c show a regular and an irregular mirror curve design of the same dimensions that generate the same **left-right-flag-design** (Figure 10.4d). Will all left-right-flag-designs be of this type? If so, what may be the reason of this symmetric invariance?



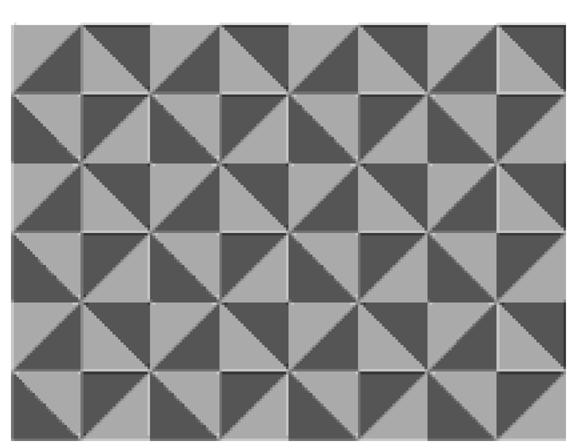
a



b



c



d

Figure 10.4

In the case one uses four colors for the left and right flags and they appear in the sequence indicated in Figure 10.5a, the mirror-curves in Figures 10.5b and c produce the four-color designs in Figure 10.5b and c. By reducing these designs modulo 2, that is by taking the third color equal to the first and the fourth equal to the second, the two-color designs in Figures 10.5d and e are generated. Will all regular mirror-curves lead to the same type of four-color design? Which symmetries and other characteristics do these **four-color-flag-designs** have in common? And what can be said about their two-color counterparts?

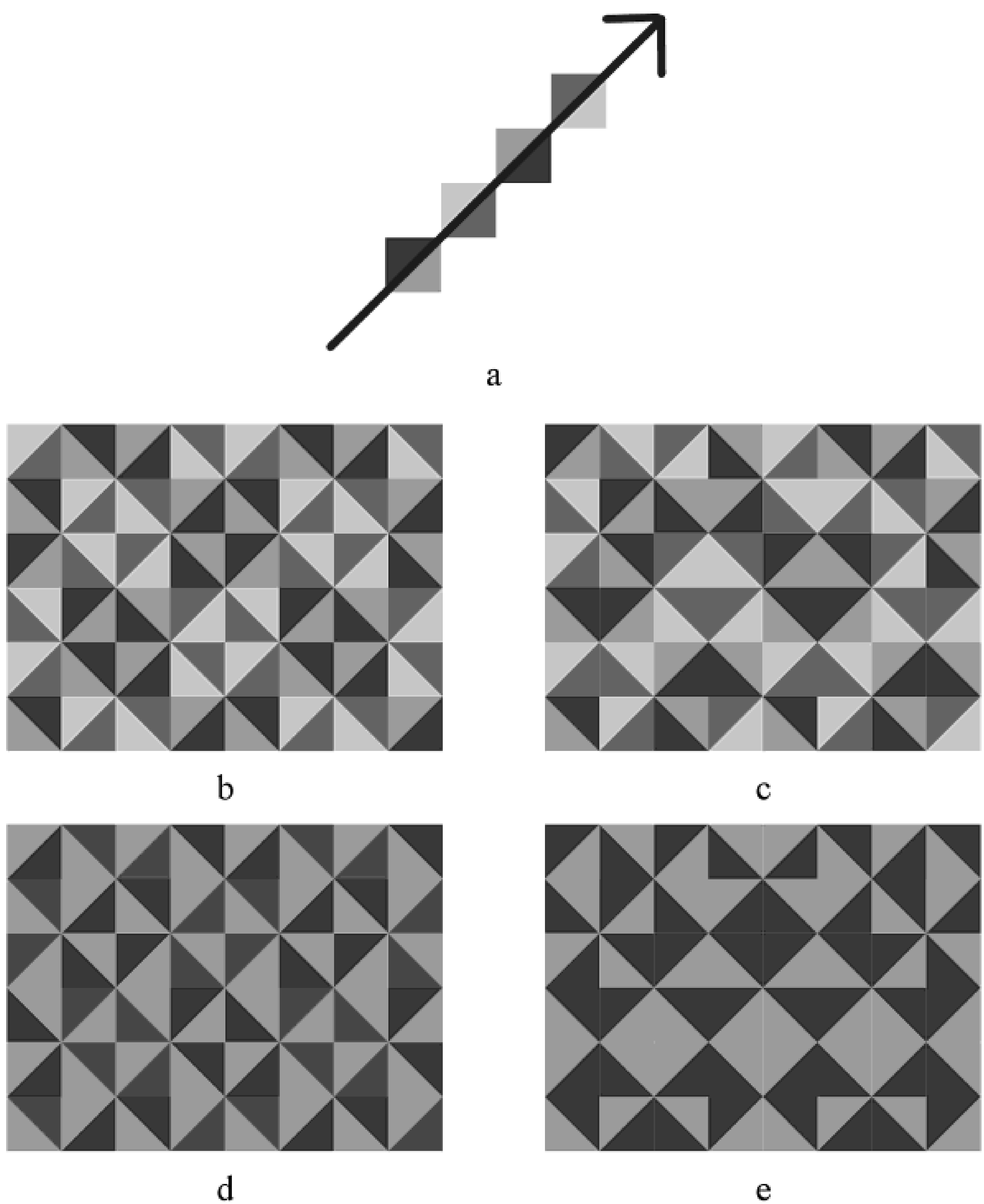
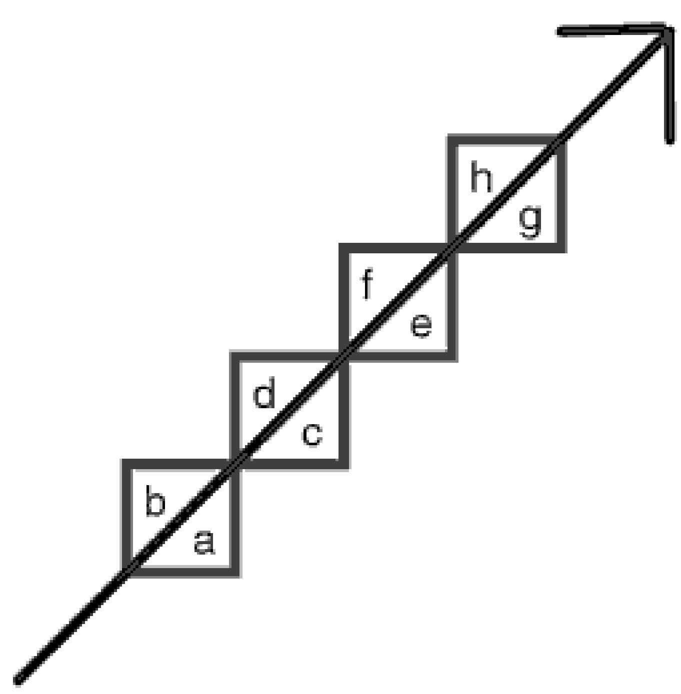
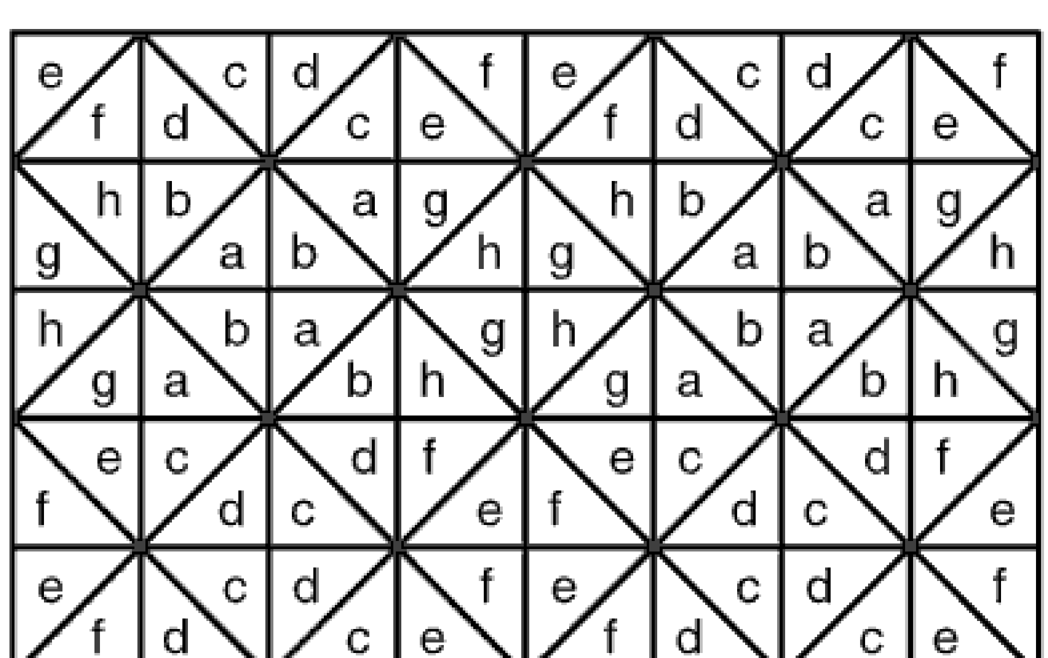


Figure 10.5

The right-flag-designs, left-right-flag-designs and the aforementioned four-color-flag-designs are all particular instances of **eight-color-flag designs** generated by coloring the unit squares through which a mirror-curve passes successively on the left and on the right with eight colors as schematically indicated in Figure 10.6a. In the case a=b=e=f, c=d=g=h, and a≠c, our eight-color designs become Lunda-designs; in the case a=c=e=g, b=d=f=h, and a≠b, they become right-flag-designs, etc. Which are the local and global symmetries displayed by these eight-color designs? For instance, Figure 10.6b and c show the eight-color designs generated by the regular and irregular mirror-curve designs in Figure 10.4b and c. Which characteristics do they have?

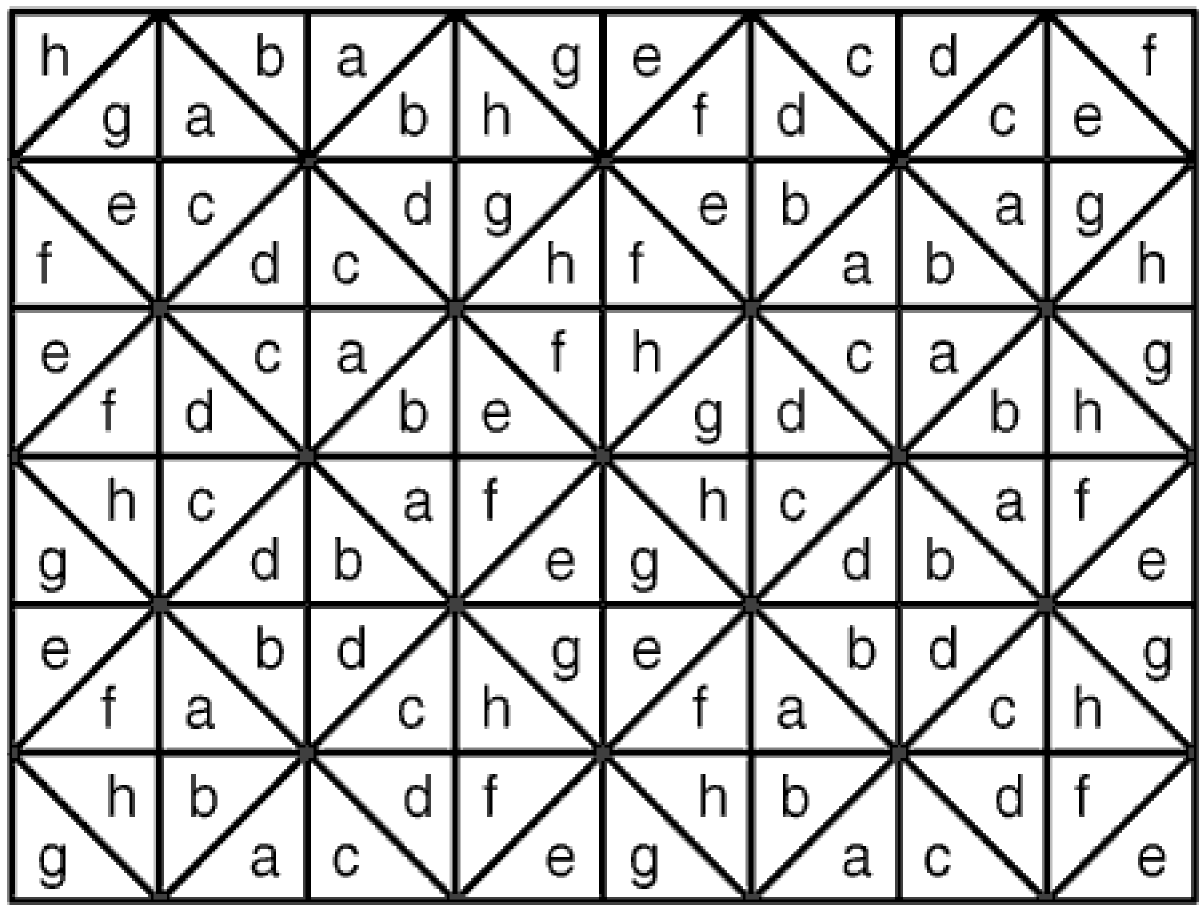


a



b

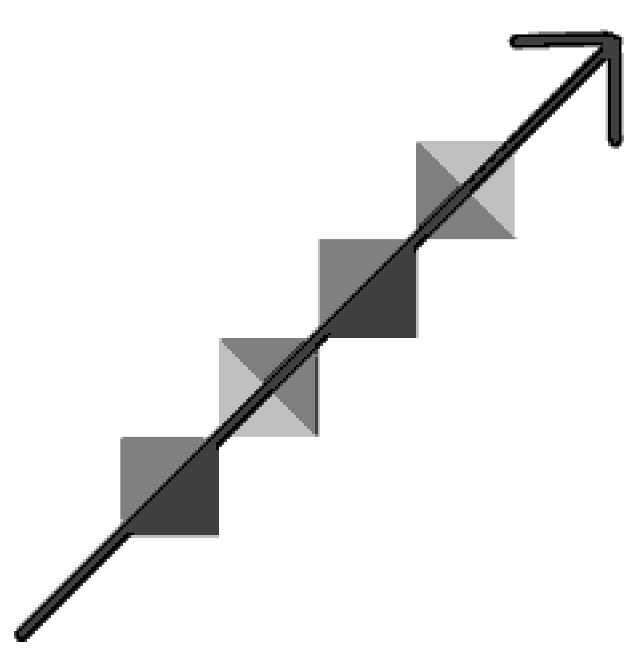
Figure 10.5 (first part)



c

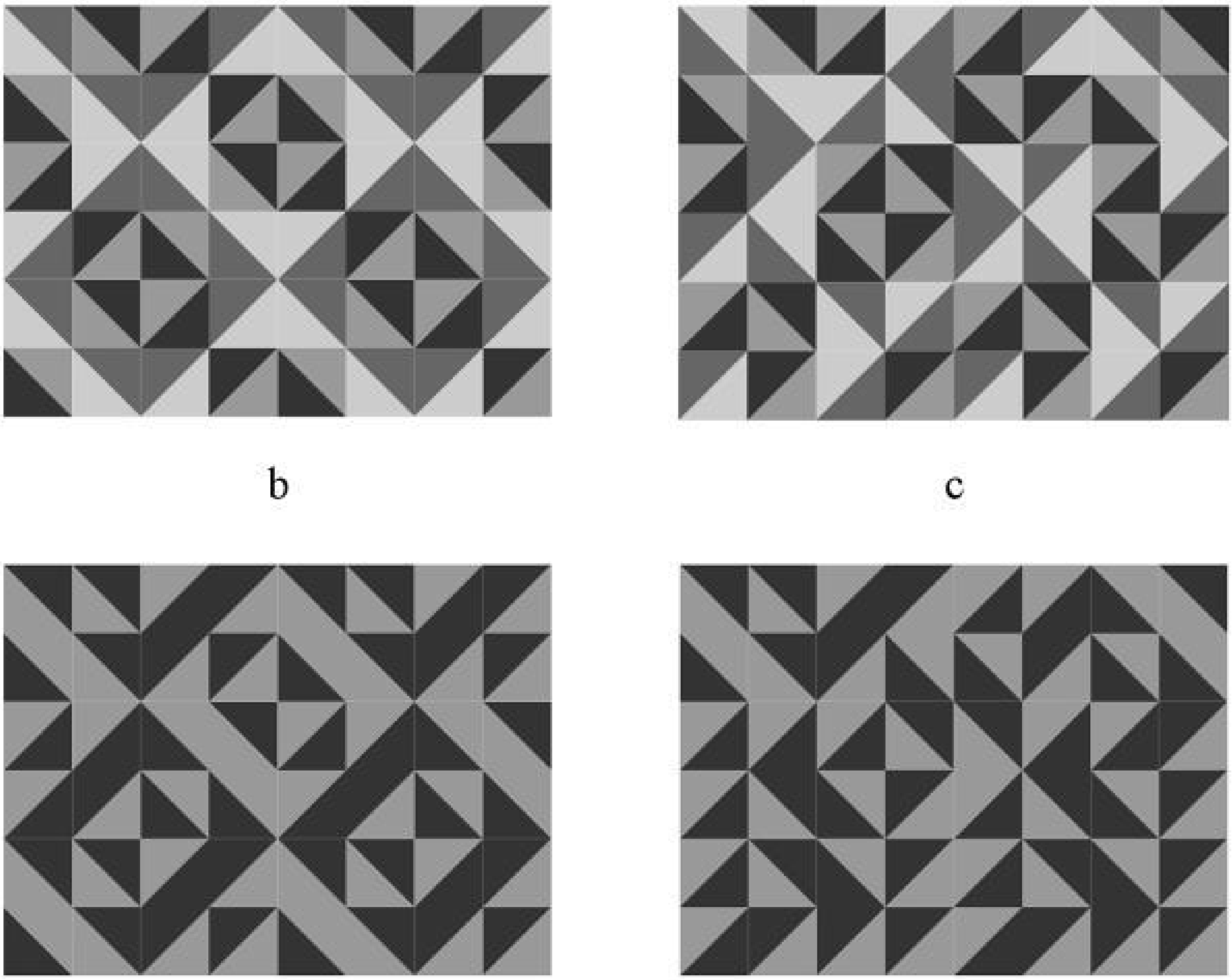
Figure 10.6 (second part)

Another type of design is produced when it is also admitted that the flags may appear not only in left and right position, but also in up and down position, in agreement, for example, with the four-color scheme displayed in Figure 10.7a. Figure 10.7b and c present the four-color designs produced in this way by the mirror-curve designs of Figure 10.4b and c. By reducing these designs modulo 2, the two-color designs in Figure 10.7d and e are generated.



a

Figure 10.7 (first part)

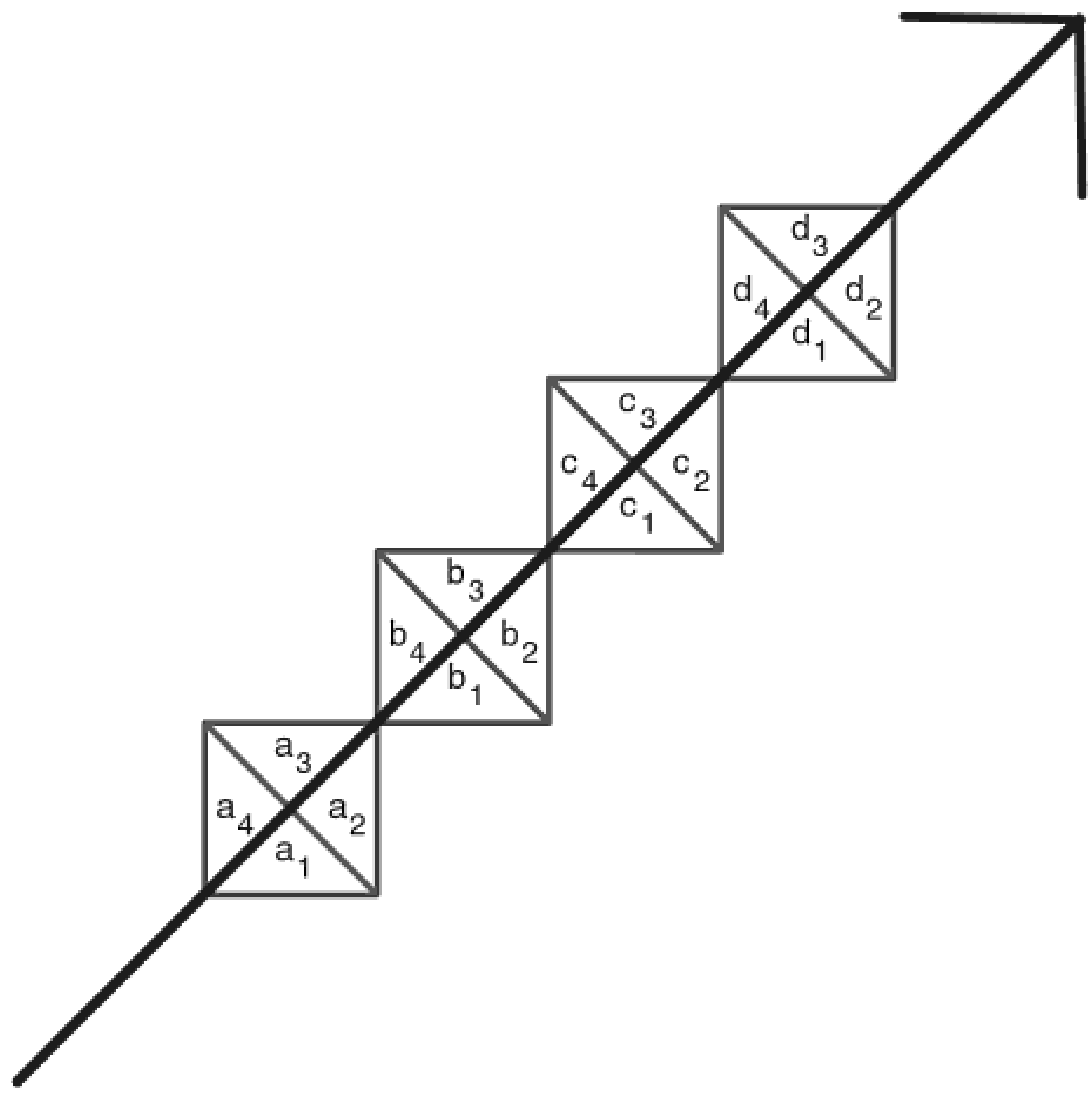


b c

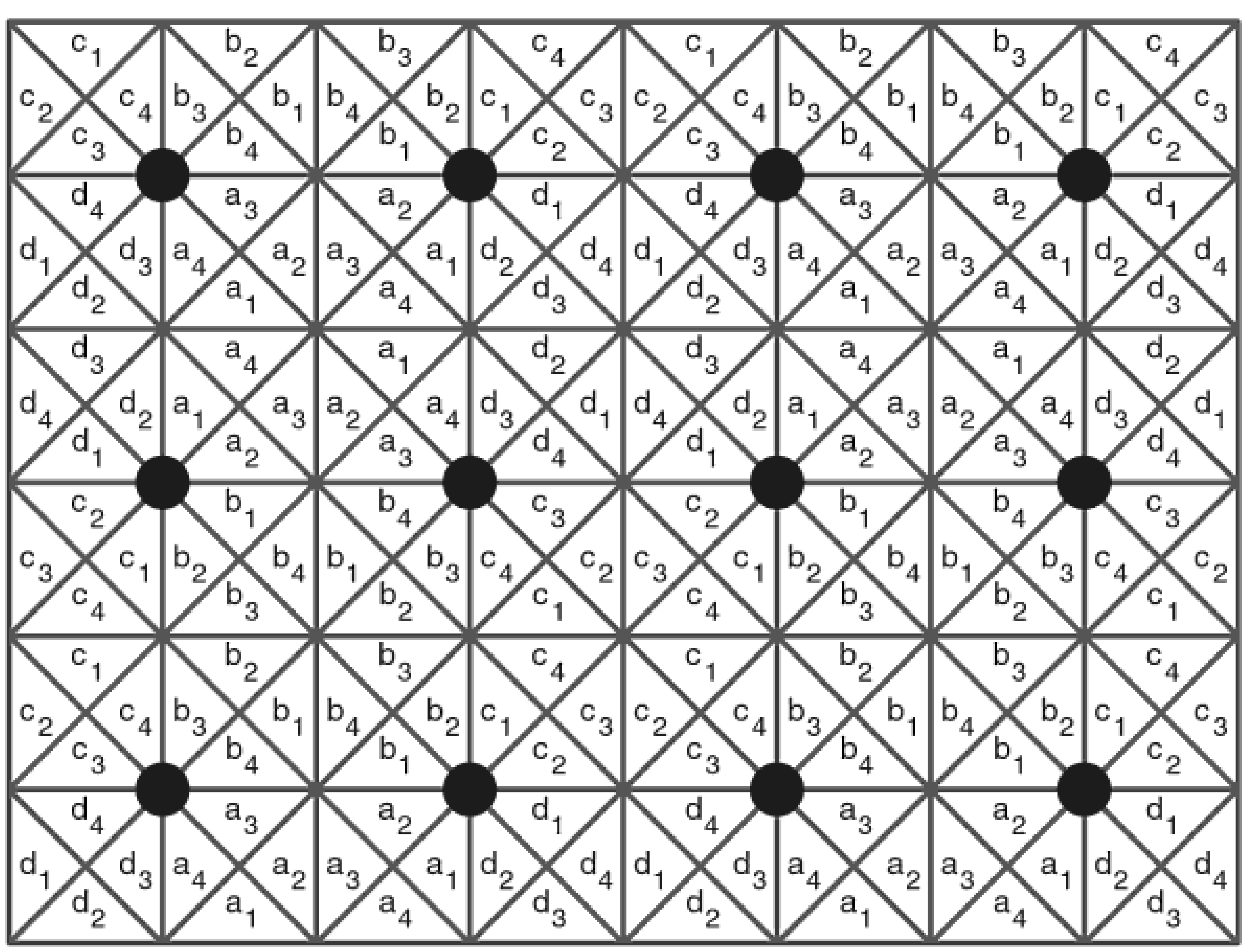
d e

Figure 10.7

These left-right, up-down flag designs together with the aforementioned generalization of eight-color flag designs, belong to a more general class of **16-colour designs** that are generated by coloring the unit squares through which a mirror-curve passes successively according to the scheme in Figure 10.8a. Figure 10.8b shows the 16-colour design produced by the reduced chased-chicken mirror-curve in Figure 10.4b. All regular mirror-curves generate similar designs. In these designs the colors in the unit squares appear in four positions, ↑ (position 1), ← (2), ↓ (3), and → (4) (see Figure 10.9a). Figure 10.9b displays the distribution of the positions 1, 2, 3 and 4 in the corresponding grid. A question open for further research is which are the common symmetry characteristics of all 16-color designs, including those generated by irregular mirror-curves. Is it possible to define these 16-color designs independently from mirror-curves, as it was possible in the particular case of Lunda-designs?

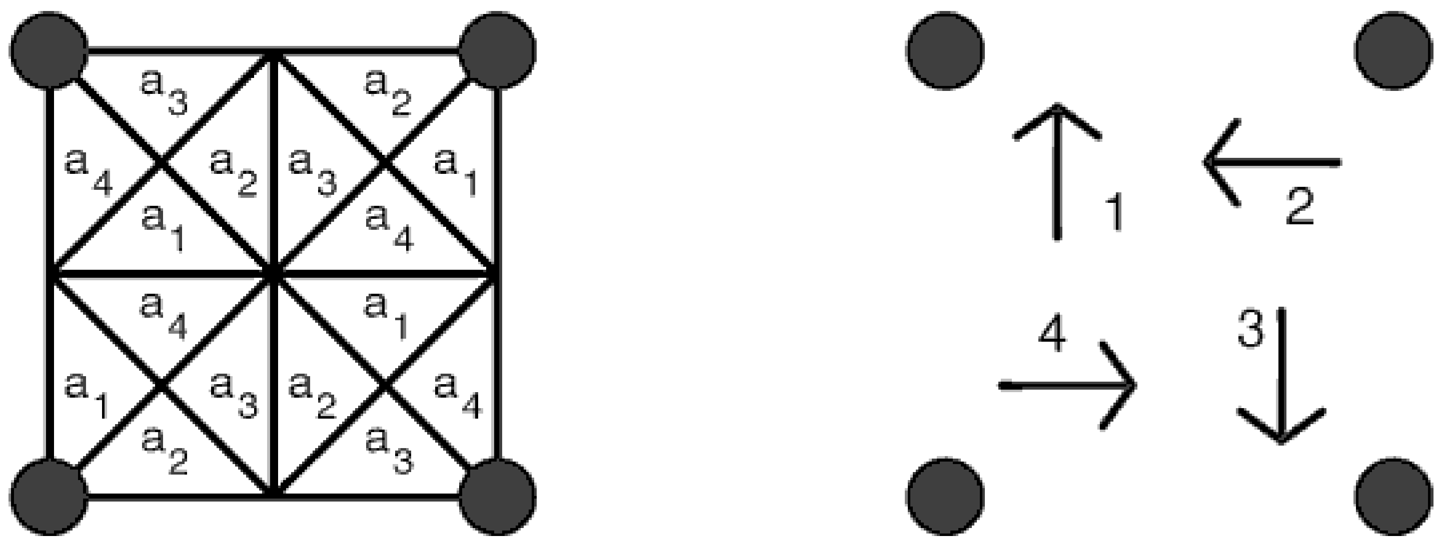


a



b

Figure 10.8



a

1	2	3	4	1	2	3	4	1	2	3	4
4	3	2	1	4	3	2	1	4	3	2	1
3	4	1	2	3	4	1	2	3	4	1	2
2	1	4	3	2	1	4	3	2	1	4	3
1	2	3	4	1	2	3	4	1	2	3	4
4	3	2	1	4	3	2	1	4	3	2	1
3	4	1	2	3	4	1	2	3	4	1	2
2	1	4	3	2	1	4	3	2	1	4	3
1	2	3	4	1	2	3	4	1	2	3	4
4	3	2	1	4	3	2	1	4	3	2	1
3	4	1	2	3	4	1	2	3	4	1	2
2	1	4	3	2	1	4	3	2	1	4	3

b

Figure 10.9

Appendix 1

PROOFS OF SOME THEOREMS CONCERNING THE RELATIONSHIP BETWEEN RECTANGLE-FILLING MIRROR CURVES AND LUNDA-DESIGNS

In this appendix we prove the theorems announced in Chapter 2 concerning the relationship between (rectangle-filling) mirror curves [or monolinear mirror lines designs] and mxn Lunda-designs.

A mxn Lunda-design is a black-and-white design on a rectangular grid RG[m,n] defined by the following characteristics:

(i) Of the two border unit squares of any grid point in the first or last row, or in the first or last column, one is always white and the other black;

(ii) Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are always black and two are white.

Theorem 1

Every (rectangle-filling) mirror curve generates a Lunda-design.

Proof

As the mirror curve traverses the rectangular grid RG[m,n] the successive unit squares it passes through are colored alternately black and white. The black-and-white design it generates in this way satisfies property (i), since the mirror curve embraces all the grid points, and when embracing a border grid point the two unit squares it then passes through on the border side are of different colors.

Let us now assume that a pair of (horizontal or vertical) neighboring grid points exist in between which there are 3 or 4 white unit squares instead of 2 (in the case of 3 or 4 black unit squares the reasoning will be the same).

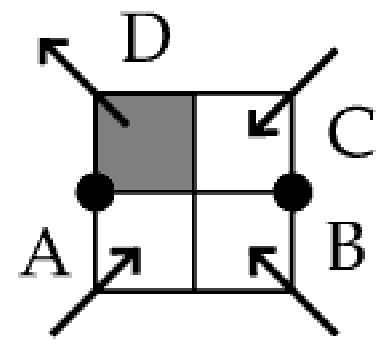


Figure A.1

We first consider the case of 3 white unit squares (see Figure A.1). When the curve enters the white unit square A it may only continue its way through the grid by entering the black unit square D, since after a white unit square a black unit square always follows. When later the curve enters the 2x2 square by the white unit square B or C, it cannot leave the 2x2 square. This means that the curve is not closed, what is in contradiction with the fact that the curve is a (rectangle-filling) mirror curve. The same reasoning applies when the curve first enters the white unit square B or C.

If the 2x2 square consisted of four white unit squares, the curve could only enter it, and not leave it (see Figure A.2). This once again contradicts the fact that the curve is closed. This completes the proof.

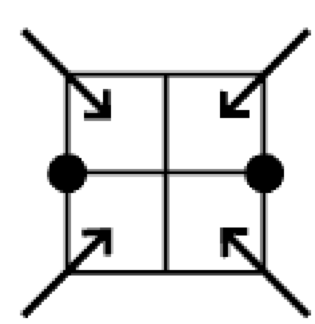


Figure A.2

Theorem 2

Any mxn Lunda-design has the following two properties:

(i) In each row there are m black and m white unit squares;
(ii) In each column there are n black and n white unit squares.

Proof:

We prove (i) row by row.

(1) In the first row (= border row) there are as many black as white unit squares, as in agreement with the definition of a Lunda-design [property (i)] each border grid point has one black and one white adjacent unit square. Moreover, since the total number of unit squares in the first row is 2m, it follows that both the number of black and the number of white unit squares are equal to *m*.

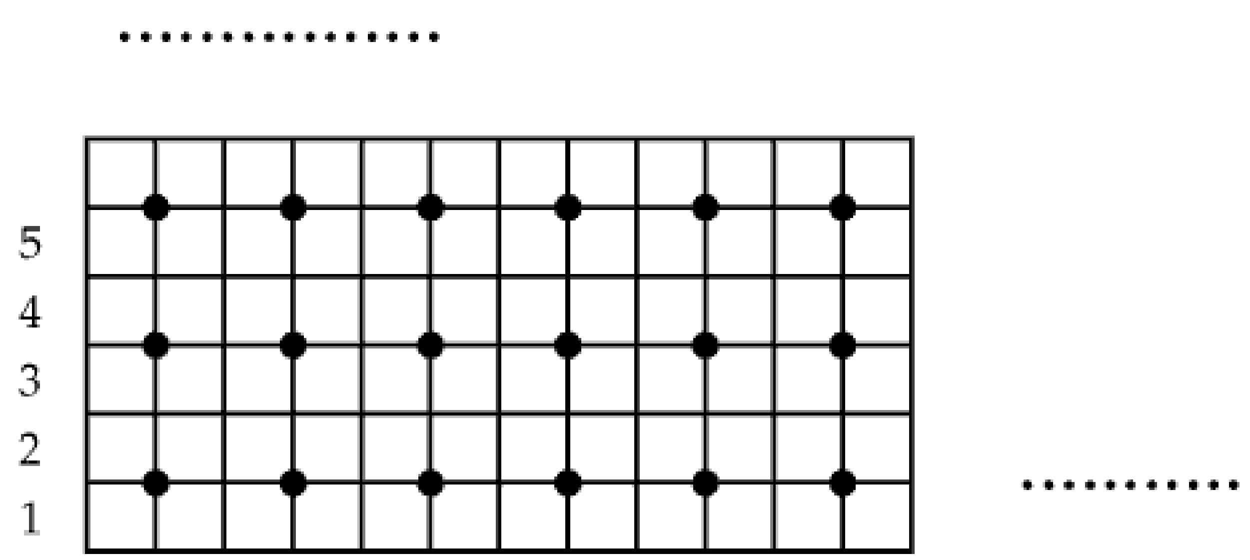


Figure A.3

(2) Consider now the first two rows of unit squares together (see Figure A.3). Between the successive grid points there are always 2 black and 2 white unit squares [property (ii)]. As there are *m* grid points on a row, there are in total 4(m-1) unit squares between them, of which half are black and half are white, i.e. 2(m-1). We have still to count the black and white unit squares in the first and last border columns, which are adjacent to grid points in the first row of grid points. Once more, according to property (i), in each of these 1x2 rectangles we have one black and one white unit square. Therefore the total number of black unit squares in the first two rows of unit squares is 2(m-1)+2 = 2m.

As, according to (1), the total number of black unit squares in the first row is *m*, it follows that the total number of black unit squares in the second row is 2m-m = m. The same is true for the white unit squares.

(3) Consider the second and third rows of unit squares together. This time these unit squares lie between vertically neighboring grid points. As there are *m* of such pairs of vertically neighboring grid points, there are in agreement with property (ii), 2m black and 2m white unit squares in the second and third row together. Since, according to (2), the number of black unit squares in the second row is *m*, it follows that the number of black unit squares in the third row is 2m-m = m. The same is true for the white unit squares.

Advancing in this way by considering pairs of successive rows of unit squares, it follows that in all rows there are exactly *m* black and *m* white unit squares. On symmetry grounds, it immediately follows that there are *n* black and *n* white unit squares in each column of unit squares. This completes the proof.

Theorem 3

Given a mxn Lunda-design, it is possible to construct a mirror curve that generates it.

Proof

The construction may be executed in the following steps:

(1) Substitution of the black-and-white 2x1 rectangles in the border rows and columns by single curve elements, as shown in Figure A.4;

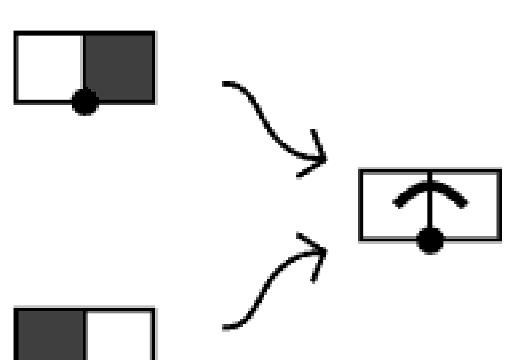


Figure A.4

(2) Substitution of the black-and-white 2x2 squares, between vertically neighboring grid points, by pairs of curve elements which are locally compatible with the coloring (see Figure A.5);

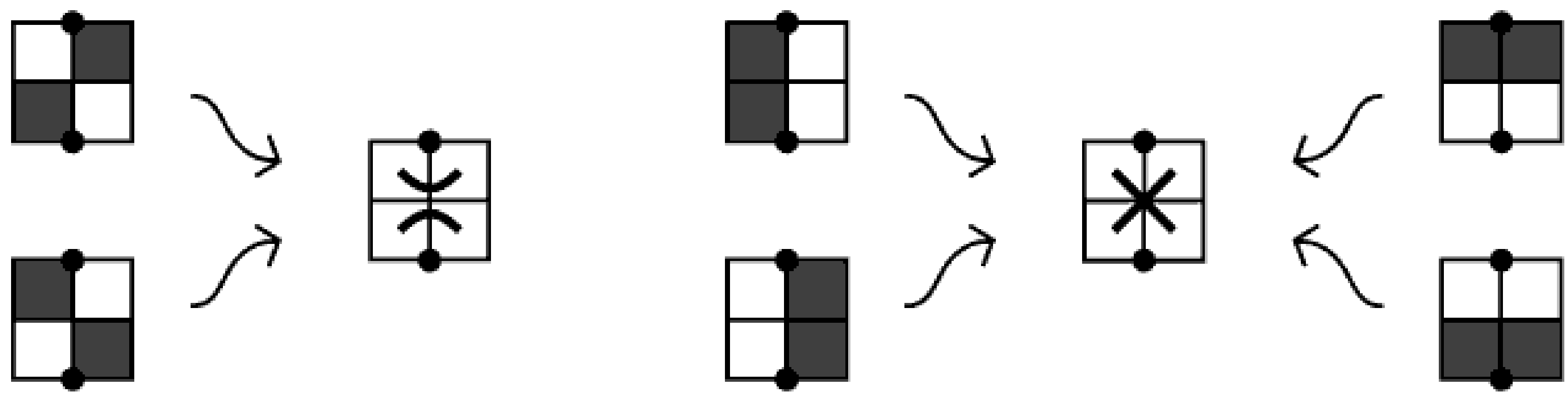


Figure A.5

(3) Substitution of the black-and-white 2x2 squares, between horizontally neighboring grid points, by pairs of curve elements which are locally compatible with the coloring (see Figure A.6).

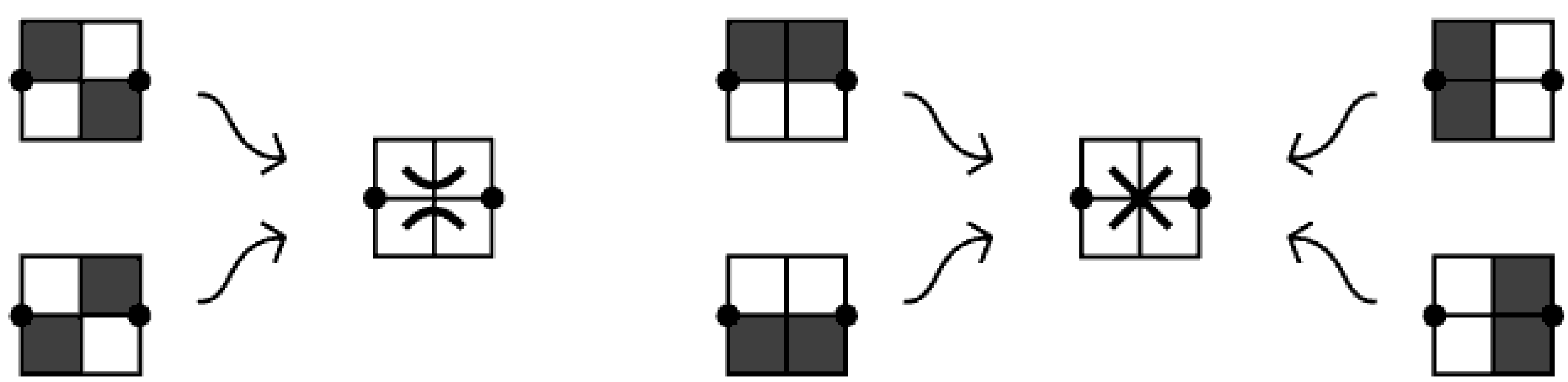
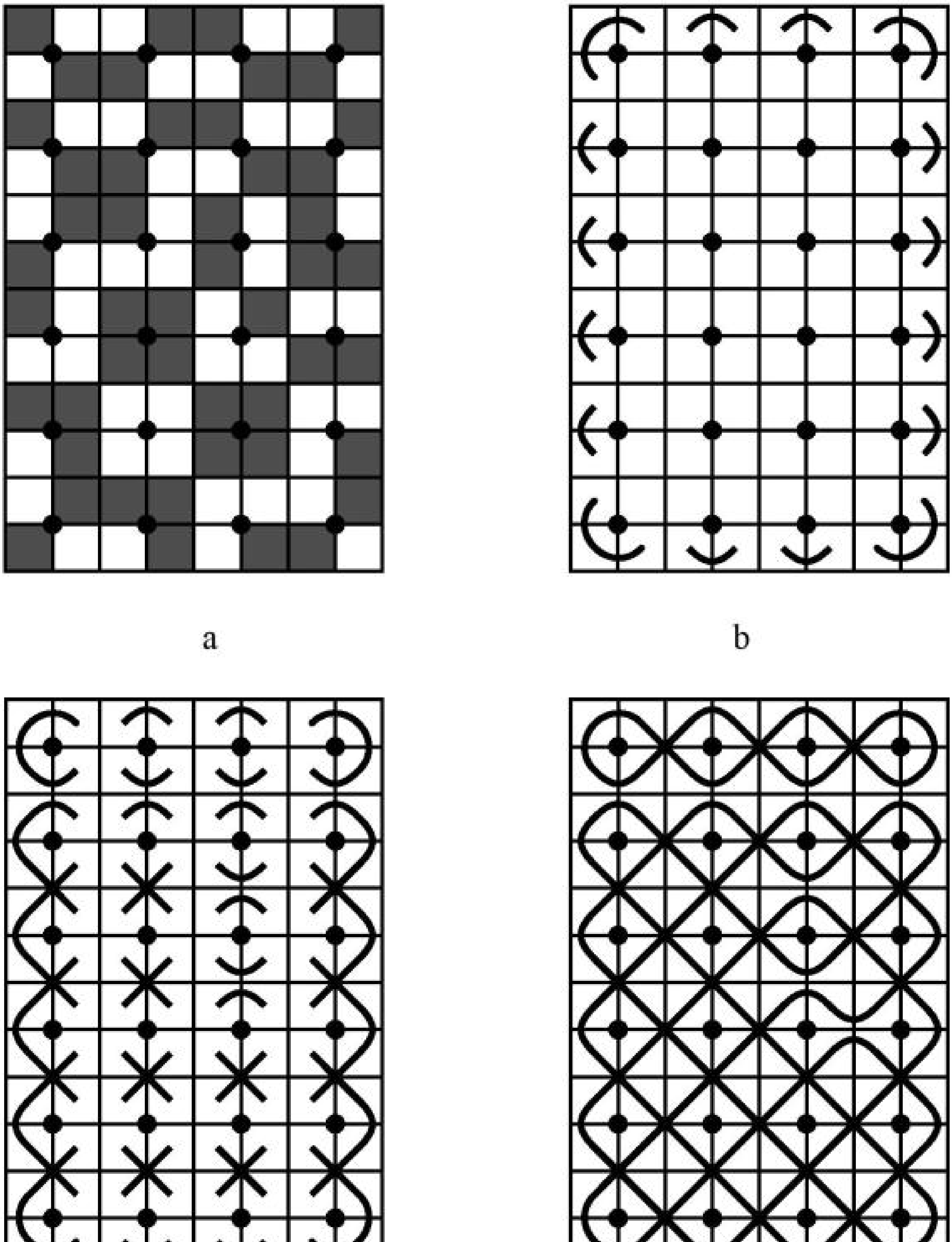


Figure A.6

Figure A.7a-d presents a concrete example of the execution of the first three steps of construction.

The total set of curve elements thus constructed constitutes one or more closed mirror curves [i.e. a monolinear or a polylinear mirror lines design, cf. chapter 1], which together embrace all grid points (four curves in the example [Figure A.7e]).

(4) If there is a crossing of two curves, it may be substituted by a couple of opposite ‘moon’ elements that is locally consistent with the coloring (see Figure A.8). In this way the two curves are transformed into one new curve and the total number of curves is reduced by one. If after this step there are still crossings of distinct curves, the step may be repeated (see steps e to f and f to g in Figure A.7).



a b

c d

Figure A.7 (first part)

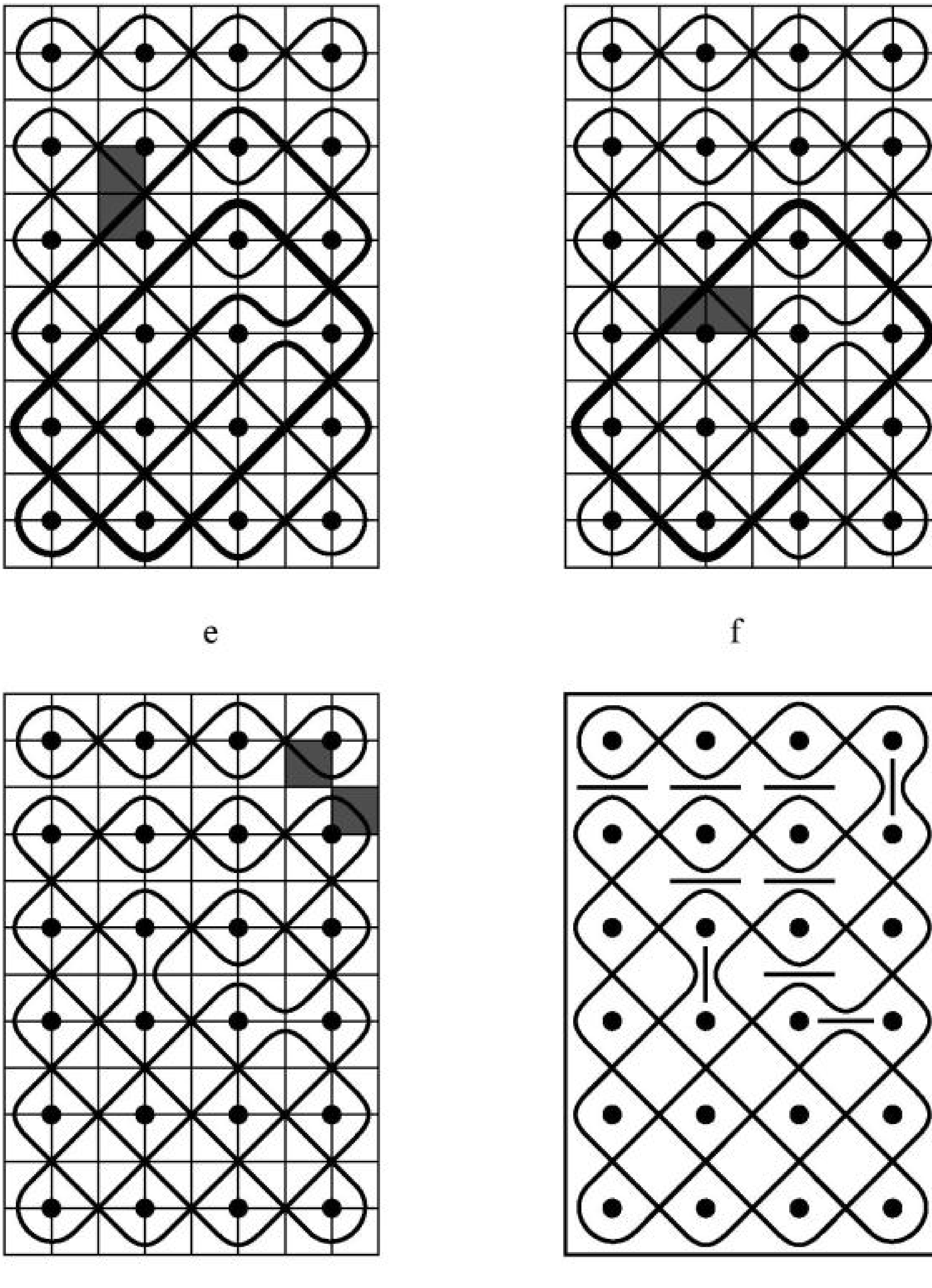


Figure A.7 (conclusion)

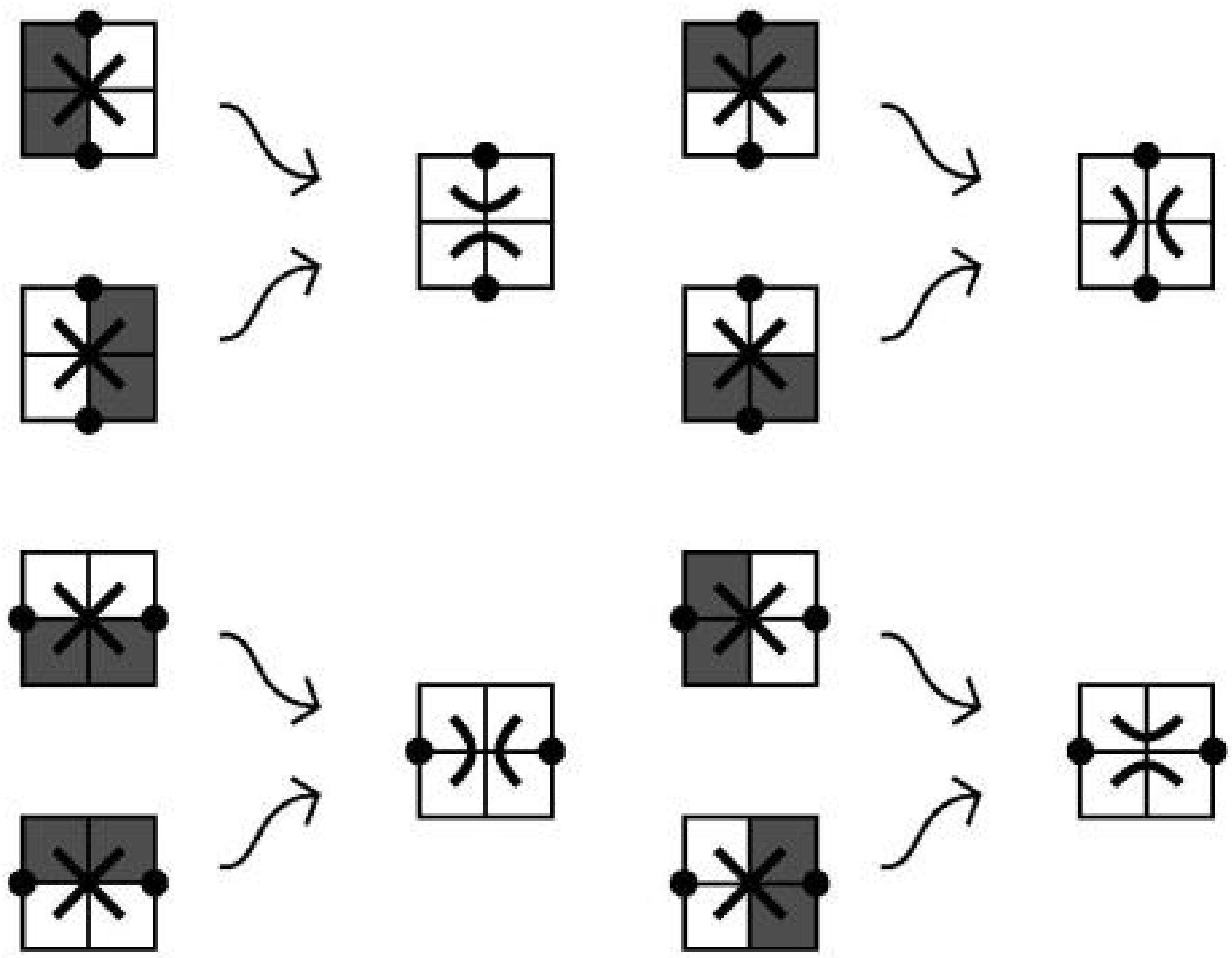


Figure A.8

(5) If there are no more crossings of distinct curves but more than one curve still exists, then there must be one or more horizontal 'kissings' of opposite 'moon' elements belonging to different curves. Such a horizontal 'kissing' has to be substituted by a vertical 'kissing' (see Figure A.9). This type of substitution does not affect the coloring of the unit squares, and reduces the total number of curves by one.

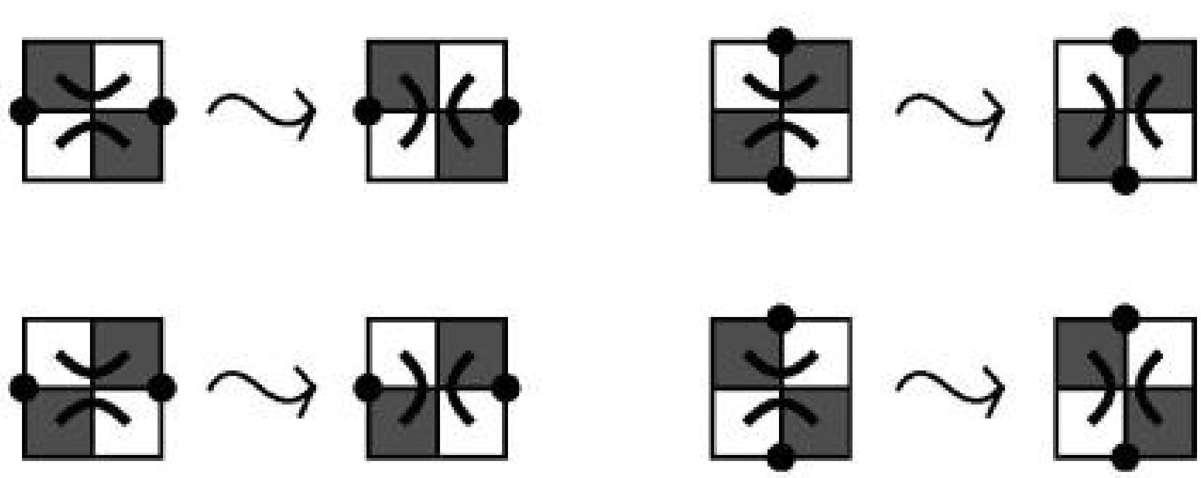


Figure A.9

Repeating step (5) as many times as necessary, the total number of curves becomes gradually reduced until a rectangle-filling mirror curve [i.e. a monolinear mirror lines design, cf. chapter 1] remains (see in the example of Figure A.7 the transition from g to h). The final position of the mirrors is in the middle of the remaining opposite 'moon' elements (see the example in Figure A.7h).

This completes the proof.

Books by Paulus Gerdes

In English:

- *Adventures in the World of Matrices*, Nova Science Publishers, New York, 2007 [Preface by Gaston N'guérékata]
- *Lunda Geometry: Mirror Curves, Designs, Knots, Polyominoes, Patterns, Symmetries*, Lulu.com, Morrisville NC, 2007, 196 pp. [First edition: Universidade Pedagógica (UP), Maputo (Mozambique), 1996, 149 pp.]
- *Drawings from Angola: Living Mathematics*, Lulu.com, Morrisville NC, 2007, 72 pp.
- *Mathematics in African History and Cultures: An annotated Bibliography* (co-author Ahmed Djebbar), New edition: Lulu.com, Morrisville NC (EUA), 2007, 430 pp. [First edition: African Mathematical Union, Cape Town (South Africa), 2004, 262 pp.] [Preface by Jan Persens, President of the African Mathematical Union]
- *African Doctorates in Mathematics: A Catalogue*, Lulu.com, Morrisville NC (EUA), 2007, 383 pp. [Preface by Mohamed Hassan, President of the African Academy of Sciences]
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- *Sona Geometry from Angola: Mathematics of an African Tradition*, Polimetrica International Science Publishers, Monza (Italy), 2006, 232 pp. [Preface by Arthur B. Powell]
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* *Awakening of Geometrical Thought in Early Culture*, MEP Press, Minneapolis MN, 2003, 200 pp. [Preface by Dirk Struik]
* *Geometry from Africa: Mathematical and Educational Explorations*, The Mathematical Association of America, Washington DC, 1999, 210 pp. [Preface by Arthur B. Powell] [*Choice Magazine*: Outstanding Academic Book 2000]
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* *Lusona – Geometrical recreations of Africa – Recréations géométriques d'Afrique*, L'Harmattan, Paris (France), 1997, 127 pp. (First edition: UP, Maputo, 1991, 118 pp.) [Preface by Aderemi Kuku, President of the African Mathematical Union]
* *Ethnomathematics and Education in Africa*, Stockholm University (Sweden), 1995, 184 pp.
* *Sona Geometry: Reflections on the sand drawing tradition of peoples of Africa south of the Equator*, UP, Maputo, 1994, Vol.1, 200 pp.
* *African Pythagoras: A study in Culture and Mathematics Education*, UP, Maputo, 1994, 103 pp.
* *Sipatsi: Technology, Art and Geometry in Inhambane* (co-author Gildo Bulafo), UP, Maputo, 1994, 102 pp.
* *Mathematics, Education and Society* (co-editors: Cristine Keitel, Alan Bishop, Peter Damerow), Science and Technology Education Document Series No. 35, UNESCO, Paris, 1989, 193 pp.
* *Marx: Let us demystify calculus*, MEP-Press, Minneapolis MN, 1985, 129 pp.

In French:

* *Les Mathématiques dans l'Histoire et les Cultures Africaines. Une Bibliographie Annotée* (co-author Ahmed Djebbar), Université de Lille & African Mathematical Union, Lille (France), 2007, 332 pp. [Preface by Jan Persens]
* *Le cercle et le carré: Créativité géométrique, artistique, et symbolique de vannières et vanniers d'Afrique, d'Amérique,*

d'Asie et d'Océanie, L'Harmattan, Paris, 2000, 301 pp. [Preface by Maurice Bazin]

* *Recréations géométriques d'Afrique – Lusona – Geometrical recreations of Africa*, L'Harmattan, Paris (França), 1997, 127 pp. (First edition: UP, Maputo, 1991, 118 pp.) [Preface by Aderemi Kuku, President of the African Mathematical Union]

* *Femmes et Géométrie en Afrique Australe*, L'Harmattan, Paris, 1996, 219 pp.

* *Une tradition géométrique en Afrique. — Les dessins sur le sable*, L'Harmattan, Paris, 1995, Vol. 1: Analyse et reconstruction, 247 pp.

* *Une tradition géométrique en Afrique. — Les dessins sur le sable*, L'Harmattan, Paris, 1995, Vol. 2: Exploration éducative et mathématique, 184 pp.

* *Une tradition géométrique en Afrique. — Les dessins sur le sable*, L'Harmattan, Paris, 1995, Vol. 3: Analyse comparative, 144 pp.

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* *Ethnomathematik dargestellt am Beispiel der Sona Geometrie*, Spektrum Verlag, Heidelberg, 1997, 436 pp. [Preface by Harald Scheid and Erhard Scholz]

* *Ethnogeometrie. Kulturanthropologische Beiträge zur Genese und Didaktik der Geometrie*, Verlag Franzbecker, Bad Salzdetfurth, 1990, 360 pp. (2^{nd} edition: 2000) [Preface by Peter Damerow]

In Portuguese:

* *Geometria e Cestaria dos Bora na Amazónia Peruana*, Lulu.com, Morrisville NC, 2007, 172 pp. [Preface by Dubner Medina Tuesta]

* *Otthava: Fazer Cestos e Geometria na Cultura Makhuwa do Nordeste de Moçambique*, Universidade Lúrio, Nampula (Mozambique) & Lulu.com, Morrisville NC, 2007, 292 pp. [Preface by Abdulcarimo Ismael, epilogue by Mateus Katupha, Former Minister of Culture, Youth and Sports of Mozambique]
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* *Aventuras no Mundo dos Triângulos*, Ministério da Educação e Cultura, Maputo, 2005, 176 pp. [Preface by Marcos Cherinda]
* *Sipatsi: Cestaria e Geometria na Cultura Tonga de Inhambane*, Moçambique Editora, Maputo & Texto Editora, Lisbon (Portugal), 2003, 176 pp. [Preface by Alcido Nguenha, Minister of Education of Mozambique]
* *Lusona: Recreações Geométricas de África*, Moçambique Editora, Maputo & Texto Editora, Lisbon, 2002, 128 pp. (First edition: Universidade Pedagógica, Maputo, 1991, 117 pp.)
* *Sipatsi: Tecnologia, Arte e Geometria em Inhambane* (co-author Gildo Bulafo), Universidade Pedagógica (UP), Maputo, 1994, 102 pp.
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* *A numeração em Moçambique: Contribuição para uma reflexão sobre cultura, língua e educação matemática*, UP, Maputo, 1993, 159 pp. (editor)
* *Pitágoras Africano: Um estudo em Cultura e Educação Matemática*, UP, Maputo, 1992, 103 pp.
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* *Cultura e o Despertar do Pensamento Geométrico*, UP, Maputo, 1992, 146 pp.
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* *Karl Marx: Arrancar o véu misterioso à matemática*, Universidade Eduardo Mondlane (UEM), Maputo, 1983, 112 pp.
* *Trigonometria, Manual da 11ª classe*, Ministério da Educação e Cultura, Maputo, 1981, 105 pp.
* *Trigonometria, Manual da 10ª classe*, Ministério da Educação e Cultura, Maputo, 1980, 188 pp.

The book “Lunda Geometry” explains how the mathematical concepts of mirror curves and Lunda-designs were discovered in the context of the author’s research of ‘sona’, illustrations traditionally made in the sand by Cokwe storytellers from eastern Angola (a region called Lunda) and neighboring regions of Congo and Zambia (Africa).

Examples of mirror curves from several cultures are presented. Lunda-designs are aesthetically attractive and display interesting symmetry properties. Examples of Lunda-patterns and Lunda-polyominoes are presented. Some generalizations of the concept of Lunda-design are discussed, like hexagonal Lunda-designs, Lunda-k-designs, Lunda-fractals, and circular Lunda-designs. Lunda-designs of Celtic knot designs are constructed.

Several chapters were published in journals like "Computers & Graphics" (Oxford), "Visual Mathematics" (Belgrade), and "Mathematics in School" (UK).

The first edition of the book was published by the ‘Universidade Pedagógica’ (Maputo, Mozambique, 1996). The new edition is expanded with two chapters, one published in the book “Symmetry 2000.”

Other books by Paulus Gerdes related to the theme of “Lunda Geometry” are:

“Geometry from Africa” (The Mathematical Association of America, Washington DC, 1999), “Sona Geometry from Angola: Mathematics of an African Tradition” (Polimetrica, Monza, 2006), “Drawings from Angola: Living Mathematics” (Lulu.com, 2007), and “Adventures in the World of Matrices” (Nova Science Publishers, New York, 2007).

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